Problem Set 5: Savings Accounts and Compound Interest

Solutions

1. Consider a savings account that earns 4% interest compounded monthly. You initially deposit $1,000.00 into the account and make no further deposits or withdrawals.
   a. Draw a flow diagram for the monthly balance in the account.

Because interest is compounded monthly, each month the balance will increase by $\frac{4}{12}\%$. The flow diagram is given in Figure 2.1.

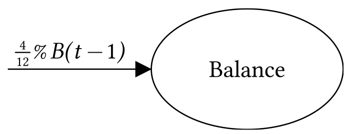

Figure 2.1: Flow diagram for Exercise 2.1.1.

   b. Find the DDS.

The inward-pointing arrow represents an increase in the balance from the previous month, so the DDS is given by

$$B(t) = B(t-1) + \tfrac{4}{12}\% B(t-1),$$

where time is in months.

   c. Determine the account balance two years from the initial deposit.

Here we can use Excel or the explicit formula. To use the explicit formula we note that the initial balance is $B(0) = 1,000,$ the monthly interest rate is $\frac{4}{12}\%,$ and the time in months is $t = 24.$ Plugging everything into the explicit formula gives

$$B(t) = \left(1 + \frac{r}{12}\right)^t B(0)$$

$$B(24) = \left(1 + \frac{0.04}{12}\right)^{24} 1,000$$

$$B(24) = 1,083.14.$$

The balance after two years, assuming no additional deposits or withdrawals, will be $1,083.14.

   d. Determine how long it will take the deposit to double.

We set the balance equal to $2,000 and solve for the required time. First we have

$$2,000 = \left(1+\frac{0.04}{12}\right)^t 1,000$$

$$2 = \left(1+\frac{0.04}{12}\right)^t.$$

From here we can use a calculator and trial-and-error to find t, or we can take the natural logarithm of both sides:

$$\ln(2) = \ln\left[\left(1+\frac{0.04}{12}\right)^t\right]$$

$$\ln(2) = t \cdot \ln\left(1+\frac{0.04}{12}\right)$$

$$0.6931 \approx t \cdot 0.003328$$

$$208.3 \approx t.$$

It would take about 208 months, or about 17.4 years for the balance in the account to double.

2. Consider a savings account that earns 6% interest compounded monthly. You make an initial deposit and then monthly deposits thereafter. If the initial deposit is \$2,000.00 and the monthly deposits are \$100:
   a. Give the flow diagram for the monthly balance.

Because the interest is compounded monthly, the account earns $\frac{6}{12}\% = 0.5\%$ interest each month. The flow diagram for the situation is given in Figure 2.2.

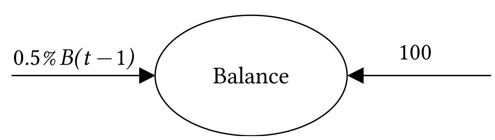

Figure 2.2: Flow diagram for Exercise 2.1.2.

   b. Find the DDS.

The inward pointing arrows represent increases in the account balance each month, so the DDS is given by

$$B(t) = B(t-1) + 0.5\%B(t-1) + 100.$$

   c. Determine the balance after 10 months using Excel.

Our Excel set-up is the same as for an exponentially growing population with stocking. We let the interest rate and monthly deposit be parameters stored in their own cells, and we refer to them with absolute addressing. Then we copy the formula down to month 10 and record the result. The result with the correct formula showing is given in Figure 2.3. The balance after ten months will be \$3,125.08.

	A		B	C
1	Exercise 2.1.2			
2				
3	Interest rate, *r* =			6.0%
4	Monthly deposit, *a* =			$ 100.00
5				
6	t		B(t)	
7	0	$	2,000.00	
8	1	=B7+($C$3/12)*B7+$C$4		
9	2	$	2,220.55	
10	3	$	2,331.65	
11	4	$	2,443.31	
12	5	$	2,555.53	
13	6	$	2,668.31	
14	7	$	2,781.65	
15	8	$	2,895.55	
16	9	$	3,010.03	
17	10	$	3,125.08	

Figure 2.3: Excel result for Exercise 2.1.2c.

d. Determine the balance after 10 months using the appropriate explicit formula.

Here we need the explicit formula for the affine model:

$$B(t) = \left(1+\tfrac{r}{12}\right)^t B(0) + a\frac{\left(1+\tfrac{r}{12}\right)^t - 1}{r/12}.$$

From the problem statement we know $B(0) = 2{,}000$, $r = 0.06$, $a = 100$, and $t = 10$. We get

$$\begin{aligned} B(10) &= \left(1+\tfrac{0.06}{12}\right)^{10} 2{,}000 + 100\frac{\left(1+\tfrac{0.06}{12}\right)^{10} - 1}{0.06/12} \\ &\approx 2{,}102.28 + 100 \cdot 10.228 \\ &\approx 3{,}125.08. \end{aligned}$$

Note that this result agrees with the one we found with Excel.

e. Suppose the initial deposit is $5,000. Determine the monthly deposit required for the balance to grow to $10,000 in two years.

Here we use Goal Seek in Excel, though we could also use the explicit formula to solve for *a*. We show the set-up in Excel just before Goal Seek is run in Figure 2.4. Note that we have copied the formula down to month 24, but we have hidden most of the rows that we do not need to see.

	A	B	C
1	Exercise 2.1.2		
2			
3	Interest rate, *r* =		6.0%
4	Monthly deposit, *a* =		$ 100.00
5			
6	t	B(t)	
7	0	$ 5,000.00	
8	1	$ 5,125.00	
9	2	$ 5,250.63	
30	23	$ 8,038.80	
31	24	$ 8,178.99	

Goal Seek
Set cell: $B$31
To value: 10,000
By changing cell: $C$4
OK Cancel

Figure 2.4: Excel set-up for Exercise 2.1.2e.

The result of a successful Goal Seek is given in Figure 2.5. We would need to deposit $171.60 per month in order to have an account balance of $10,000 in 24 months.

	A	B	C
1	Exercise 2.1.2		
2			
3	Interest rate, *r* =		6.0%
4	Monthly deposit, *a* =		$ 171.60
5			
6	t	B(t)	
7	0	$ 5,000.00	
8	1	$ 5,196.60	
9	2	$ 5,394.19	
30	23	$ 9,779.50	
31	24	$ 10,000.00	

Figure 2.5: Excel output for Exercise 2.1.2e.

3. Consider a savings account that earns 5% interest compounded monthly. Suppose you will withdraw $500 per month from the account.
   a. Find the DDS for the balance.

Interest serves to increase the account balance while withdrawals decrease it. The DDS is given by $B(t) = B(t-1) + \frac{5}{12}\% B(t-1) - 500,$ where the time units are months.

   b. Find the equilibrium value for the balance.

Here we need to find B^* such that $B^* = B^* + \frac{5}{12}\% B^* - 500$. We have

$$\begin{aligned} 0 &= \tfrac{5}{12}\% B^* - 500 \\ 500 &= 0.0041667 B^* \\ 120,000 &= B^*. \end{aligned}$$

The equilibrium balance for the account is \$120,000.

c. Determine the stability of the equilibrium value.

Since this situation is equivalent to an exponentially increasing population with harvesting, we know that the equilibrium value is unstable. To confirm this we create a graph in Excel that shows the account balance over time for several different initial balances near \$120,000. Figure 2.6 shows the result confirming that the equilibrium is unstable since all balances that start off of \$120,000 are moving further away from it.

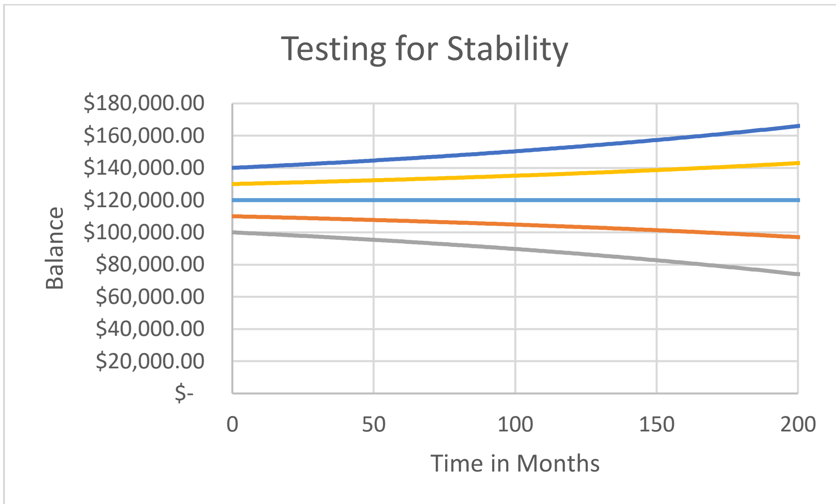

Figure 2.6: Excel graph testing for stability in Exercise 2.1.3.

d. Interpret the meaning of the equilibrium value in the context of a savings account balance.

The equilibrium value of \$120,000 is the minimum balance we would need in the account in order to withdraw \$500 per month indefinitely without running out of money.

4. Decide on something you would like to save up for, e.g., a car, a down payment on a house, a charitable contribution, etc., and choose your time horizon, i.e. how long you want it to take. Find an online savings or money market account and use the interest rate given. Assume monthly compounding. If you can afford to deposit 10% of your goal initially, determine the monthly deposit required for you to reach your goal in the time you choose.

Suppose the goal is to save \$100,000 in 10 years and that an online savings account currently pays 1% interest compounded monthly. If we can afford to deposit 10% of the goal in an account initially, then our initial deposit will be \$10,000 in this example. We could use Goal Seek in Excel to find the required monthly deposit, but here we show how to do it with the explicit formula. We have

$$B(t) = \left(1+\tfrac{r}{12}\right)^t B(0) + a\frac{\left(1+\tfrac{r}{12}\right)^t - 1}{r/12}.$$

Solving for a gives us

$$a = \frac{B(t) - \left(1+\frac{r}{12}\right)^{t} B(0)}{\left(\dfrac{\left(1+\frac{r}{12}\right)^{t} - 1}{r/12}\right)}.$$

Next we plug in all known parameters and use a calculator to find the result:

$$\begin{aligned} a &= \frac{B(120) - \left(1+\frac{0.01}{12}\right)^{120} 10{,}000}{\left(\dfrac{\left(1+\frac{0.01}{12}\right)^{120} - 1}{0.01/12}\right)} \\ &= \frac{100{,}000 - \left(1+\frac{0.01}{12}\right)^{120} 10{,}000}{\left(\dfrac{\left(1+\frac{0.01}{12}\right)^{120} - 1}{0.01/12}\right)} \\ &\approx \frac{88{,}948.75}{126.1499} \\ &\approx 705.10. \end{aligned}$$

We would need to deposit \$705.10 per month to reach \$100,000 in 10 years if we initially deposit \$10,000 and have an account earning 1% compounded monthly.

5. The "Rule of 70" is a useful financial rule of thumb that estimates the doubling time for an investment. The Rule of 70 says that if r is the interest rate (given as a percent) for a savings account (or rate of return for an investment), then the time it takes for an initial deposit to double, assuming no further deposits or withdrawals, is approximately $\dfrac{70}{r}$ in years.
    a. Compare what the rule estimates for doubling time with the actual doubling time for $r = 10\%$.

Rule of 70 estimate: $\dfrac{70}{10} = 7$ years.

Actual: Use natural logarithms to solve for the doubling time. (We could also use trial and error with a calculator.) Here we assume that the interest is compounded annually and that our time units are years.

$$
\begin{aligned}
B(t) &= (1+r)^t B(0) \\
2\,\mathrm{B}(0) &= (1+0.10)^t B(0) \\
2 &= (1.10)^t \\
\ln 2 &= \ln\left[(1.10)^t\right] \\
0.6931 &\approx t \cdot \ln(1.10) \\
0.6931 &\approx t \cdot 0.0953 \\
7.27 &\approx t.
\end{aligned}
$$

The actual doubling time is about 7.3 years, which is very close to the estimate given by the Rule of 70. Note that the result is independent of the initial deposit since the initial deposit ends up canceling. We could have used any value for $B(0)$ and we would get the same doubling time.

b. Compare what the rule estimates for doubling time with the actual doubling time for $r = 7\%$.

Rule of 70 estimate: $\frac{70}{7} = 10$ years.

Actual: Use natural logarithms to solve for the doubling time. (We could also use trial and error with a calculator.) Here we assume that the interest is compounded annually and that our time units are years.

$$
\begin{aligned}
B(t) &= (1+r)^t B(0) \\
2\,\mathrm{B}(0) &= (1+0.07)^t B(0) \\
2 &= (1.07)^t \\
\ln 2 &= \ln\left[(1.07)^t\right] \\
0.6931 &\approx t \cdot \ln(1.07) \\
0.6931 &\approx t \cdot 0.0677 \\
10.24 &\approx t.
\end{aligned}
$$

The actual doubling time is about 10.2 years, which is very close to the estimate given by the Rule of 70.

c. Compare what the rule estimates for doubling time with the actual doubling time for $r = 5\%$.

Rule of 70 estimate: $\frac{70}{5} = 14$ years.

Actual: Use natural logarithms to solve for the doubling time. (We could also use trial and error with a calculator.) Here we assume that the interest is compounded annually and that our time units are years.

$$
\begin{aligned}
B(t) &= (1+r)^t\, B(0) \\
2\,\mathrm{B}(0) &= (1+0.05)^t\, B(0) \\
2 &= (1.05)^t \\
\ln 2 &= \ln\left[(1.05)^t\right] \\
0.6931 &\approx t \cdot \ln(1.05) \\
0.6931 &\approx t \cdot 0.0488 \\
14.21 &\approx t.
\end{aligned}
$$

The actual doubling time is about 14.2 years, which is very close to the estimate given by the Rule of 70.

d. Compare what the rule estimates for doubling time with the actual doubling time for $r = 3\%$.

Rule of 70 estimate: $\dfrac{70}{3} \approx 23.33$ years.

Actual: Use natural logarithms to solve for the doubling time. (We could also use trial and error with a calculator.) Here we assume that the interest is compounded annually and that our time units are years.

$$
\begin{aligned}
B(t) &= (1+r)^t\, B(0) \\
2\,\mathrm{B}(0) &= (1+0.03)^t\, B(0) \\
2 &= (1.03)^t \\
\ln 2 &= \ln\left[(1.03)^t\right] \\
0.6931 &\approx t \cdot \ln(1.03) \\
0.6931 &\approx t \cdot 0.0296 \\
23.45 &\approx t.
\end{aligned}
$$

The actual doubling time is about 23.5 years, which is very close to the estimate given by the Rule of 70.

Overall the Rule of 70 seems to give a very good estimate for doubling times for a fairly wide range of reasonable interest rates.

6. *Extension*: Use the fact that $\ln(1+r) \approx r$ for r near 0 to derive the Rule of 70.

To solve for a doubling time in general we have

$$B(t) = (1+r)^t B(0)$$
$$2B(0) = (1+r)^t B(0)$$
$$2 = (1+r)^t$$
$$\ln 2 = \ln\left[(1+r)^t\right]$$
$$\ln 2 = t \cdot \ln(1+r)$$
$$\frac{\ln 2}{\ln(1+r)} = t.$$

Since $\ln(1+r) \approx r$ we have $t \approx \dfrac{\ln 2}{r}$, where r is given as a decimal. Thus we have

$$t \approx \frac{0.6931}{r} \approx \frac{0.70}{r}.$$

Multiplying numerator and denominator by 100 gives $t \approx \dfrac{70}{100r}$, or

$$t \approx \frac{70}{r},$$

where r is given as a percent.

7. Consider a savings account that earns 3% interest compounded monthly. Suppose you initially deposit \$200 and subsequently deposit \$50.00 per month.
   a. Use Excel to find the balance in the account after 4 years.

We use our savings account spreadsheet where time is in months, and we need to drag the formula down to month 48 to find the balance in 4 years. The Excel formula and results are shown in Figure 2.7. Note that most of the rows are hidden. In four years the balance will be \$2,772.03.

	A	B	C
1	Exercise 2.1.7		
2			
3	Interest rate, *r* =		3.0%
4	Monthly deposit, *a* =		\$ 50.00
5			
6	t	B(t)	
7	0	\$ 200.00	
8	1	=B7+(\$C\$3/12)*B7+\$C\$4	
9	2	\$ 301.13	
54	47	\$ 2,715.24	
55	48	\$ 2,772.03	

Figure 2.7: Excel set-up and results for Exercise 2.1.7.

   b. Use the appropriate explicit formula to find the same balance.

To find the same balance using the explicit formula we plug all known parameters into the formula:

$$B(t) = \left(1+\tfrac{r}{12}\right)^t B(0) + a\frac{\left(1+\tfrac{r}{12}\right)^t - 1}{r/12}$$

$$B(48) = \left(1+\tfrac{0.03}{12}\right)^{48} 200 + 50\frac{\left(1+\tfrac{0.03}{12}\right)^{48} - 1}{0.03/12}$$

$$B(48) \approx 225.47 + 50 \cdot 50.9312$$

$$B(48) \approx 2,772.03.$$

We get the same result as with Excel.

c. Over the entire four years, how much did you actually deposit into the account?

We made an initial deposit of \$200, plus 48 deposits of \$50.00 each. Thus the total deposited was $\$200 + 48 \times \$50 = \$2,600.00$.

d. By comparing your answer in c. to the account balance after 4 years, determine how much total interest you earned over the 4 years.

Since the account balance after four years is \$2,772.03 but we only deposited \$2,600.00, we must have earned \$2,772.03 - \$2,600.00 = \$172.03 in interest.

8. Suppose you have an account that earns 6% interest. Initially you deposit \$1,000 into the account.
   a. Find the balance after one year if the 6% is paid annually.

If the interest is paid annually, our time units are years and we have

$$\begin{aligned} B(1) &= (1+.06)^1 1,000 \\ &= 1,060. \end{aligned}$$

The balance after one year will be \$1,060.00.

b. Find the balance after one year if the 6% is compounded monthly.

If the interest is compounded monthly, our time units are months and we have

$$\begin{aligned} B(12) &= \left(1+\tfrac{.06}{12}\right)^{12} 1,000 \\ &= 1,061.68. \end{aligned}$$

The balance after one year will be \$1,061.68. Monthly compounding has earned us an additional \$1.68 in interest over annual compounding.

c. When compounding monthly, the actual monthly rate is found by taking 6%/12. What is the analogous daily rate if interest is compounded daily?

If interest is to be compounded daily, we should earn 6%/365 in interest each day.

d. Find the balance after one year if the 6% is compounded daily.

If the interest is compounded daily, our time units are days and we have

$$\begin{aligned} B(365) &= \left(1+\tfrac{.06}{365}\right)^{365} 1{,}000 \\ &= 1{,}061.83. \end{aligned}$$

The balance after one year will be \$1,061.83. Daily compounding nets us an additional \$0.15 in interest over monthly compounding.

9. *Extension:* Suppose you have found the world's greatest savings account, and it pays 100% interest. You deposit \$1.00 into the account initially and make no further deposits or withdrawals.
   a. Find the balance after one year if interest is paid annually.

$$\begin{aligned} B(1) &= (1+1)^{1} 1.00 \\ &= 2. \end{aligned}$$

The balance after one year will be \$2.00.

   b. Find the balance after one year if interest is compounded monthly.

With monthly compounding we use months as the time unit.

$$\begin{aligned} B(12) &= \left(1+\tfrac{1}{12}\right)^{12} 1.00 \\ &\approx 2.613. \end{aligned}$$

After one year the account balance will be about \$2.61.

   c. Find the balance after one year if interest is compounded daily.

With daily compounding we use days as the time unit.

$$\begin{aligned} B(365) &= \left(1+\tfrac{1}{365}\right)^{365} 1.00 \\ &\approx 2.7146. \end{aligned}$$

After one year the account balance will be about \$2.71.

   d. Find the balance after one year if interest is compounded hourly.

With hourly compounding we use hours as the time unit, and we need to know that there are $24 \times 365 = 8,760$ hours in a year.

$$\begin{aligned} B(8,760) &= \left(1+\tfrac{1}{8,760}\right)^{8,760} 1.00 \\ &\approx 2.7181. \end{aligned}$$

After one year the account balance will be about \$2.72.

   e. Find the balance after one year if interest is compounded every second.

With compounding every second we use seconds as the time unit, and we need to know that there are $60 \times 60 \times 24 \times 365 = 31,536,000$ seconds in a year.

$$\begin{aligned} B(31,536,000) &= \left(1+\tfrac{1}{31,536,000}\right)^{31,536,000} 1.00 \\ &\approx 2.71828. \end{aligned}$$

After one year the account balance will be about \$2.72.

f. Does there seem to be a limit on how high your balance can grow due to more frequent compounding? Do you recognize this limit?

While it is true that more frequent compounding is better for the savings account, there does seem to be a limit to how much more frequent compounding can help. In this example the limit is about an additional \$.72 over annual compounding. Note that we see almost all of the gains that more frequent compounding can provide once we are at daily compounding.

If we look at the last account balance when interest was compounded every second, we have the number 2.71828. This number is the first 6 digits of the irrational number e.

10. Though individual needs of course vary, a rough estimate for the amount of savings someone needs to live on in retirement is \$1,000,000. Determine how much per month a person would need to save in order to retire with a nest egg of \$1 million. Assume that the person retires at age 65 and has an account that earns the equivalent of 9% compounded monthly.

a. Carry out the calculations if the person starts saving at age 25, 30, 35, 40, 45, 50, 55, and 60. To organize your results, fill in a table like Text Table 2.1.

Age When Savings Begins	Years Until Retirement	Monthly Contribution Required for \$1 Million Goal
25	40	\$286.45
30	35	\$435.94
35	30	\$670.98
40	25	\$1,051.50
45	20	\$1,697.73
50	15	\$2,889.85
55	10	\$5,466.09
60	5	\$13,609.73

Text Table 2.1: Monthly contributions required to save \$1 million by age 65 for different starting ages.

Here we can use Goal Seek on the monthly payment or we can use the explicit formula to solve for a. Since no initial balance is specified, we assume that the initial balance is \$0.00 in all cases. Thus in general we can solve for a as follows:

$$B(t) = \left(1+\tfrac{r}{12}\right)^t B(0) + a\frac{\left(1+\tfrac{r}{12}\right)^t - 1}{r/12}$$

$$B(t) = a\frac{\left(1+\tfrac{r}{12}\right)^t - 1}{r/12}$$

$$\frac{B(t)}{\left[\dfrac{\left(1+\tfrac{r}{12}\right)^t - 1}{r/12}\right]} = a.$$

If the person starts saving at age 25, then saving takes place over a period of 40 years, or 480 months. Since the goal is an account balance of $1,000,000.00, we must have $B(480) = 1,000,000.00$. To find a we plug in the known parameters and use our calculator to get the result:

$$\begin{aligned} a &= \frac{B(t)}{\left[\dfrac{\left(1+\tfrac{r}{12}\right)^t - 1}{r/12}\right]} \\ &= \frac{1,000,000}{\left[\dfrac{\left(1+\tfrac{0.09}{12}\right)^{480} - 1}{0.09/12}\right]} \\ &\approx 213.61. \end{aligned}$$

If the person starts saving at age 25, a deposit of $213.61 into the account each month will be sufficient to reach $1 million by age 65.

If the person starts saving at age 30, then saving takes place over a period of 35 years, or 420 months. To find a we plug in the known parameters and use our calculator to get the result:

$$\begin{aligned} a &= \frac{B(t)}{\left[\dfrac{\left(1+\tfrac{r}{12}\right)^t - 1}{r/12}\right]} \\ &= \frac{1,000,000}{\left[\dfrac{\left(1+\tfrac{0.09}{12}\right)^{420} - 1}{0.09/12}\right]} \\ &\approx 339.93. \end{aligned}$$

If the person starts saving at age 30, a deposit of $339.93 into the account each month will be sufficient to reach $1 million by age 65.

The calculations for the remaining starting ages are all similar, and in each case the monthly deposit required increases as the age when saving begins increases.

b. What's the moral of the story?

Start early! The two examples worked out above show that delaying saving by even 5 years caused the required monthly deposit to increase by about 60%.

11. The $1,000,000.00 goal we used for our retirement savings plan is based on someone who needs to generate $40,000 a year to live on in retirement. Some people may need less income and others more. Take a few moments to decide how much income you would like to have to live on during your retirement years.
    a. A reasonable rule of thumb is that you need to accumulate roughly 25 times your desired income in order to generate that income without dipping into your savings. Based on your income estimate and this rule of thumb, how much will you need to accumulate by the time you retire?

Suppose we want an income of $55,000 per year in retirement. Then using the rule of thumb above, we would need to save $25\times\$55,000=\$1,375,000$ by the time we retire.

    b. At what age would you like to retire?

Suppose we want to retire by age 60.

    c. At what age do you foresee being able to start saving for retirement?

Assume we can start saving when we are 27 years old.

    d. Assuming that your retirement account will earn the equivalent of 9% interest compounded monthly, determine how much you will need to save each month during your career.

Starting at age 27 and retiring at age 60 means we will be saving for 33 years, or 396 months. We could use Goal Seek to find the required monthly deposit, but proceeding as in Exercise 2.1.10 note that with no initial deposit we can find a as follows:

$$\begin{aligned} a &= \frac{B(t)}{\left[\dfrac{\left(1+\frac{r}{12}\right)^t-1}{r/12}\right]} \\ &= \frac{1,375,000}{\left[\dfrac{\left(1+\frac{0.09}{12}\right)^{396}-1}{0.09/12}\right]} \\ &\approx 564.23. \end{aligned}$$

We would have to save $564.23 per month.