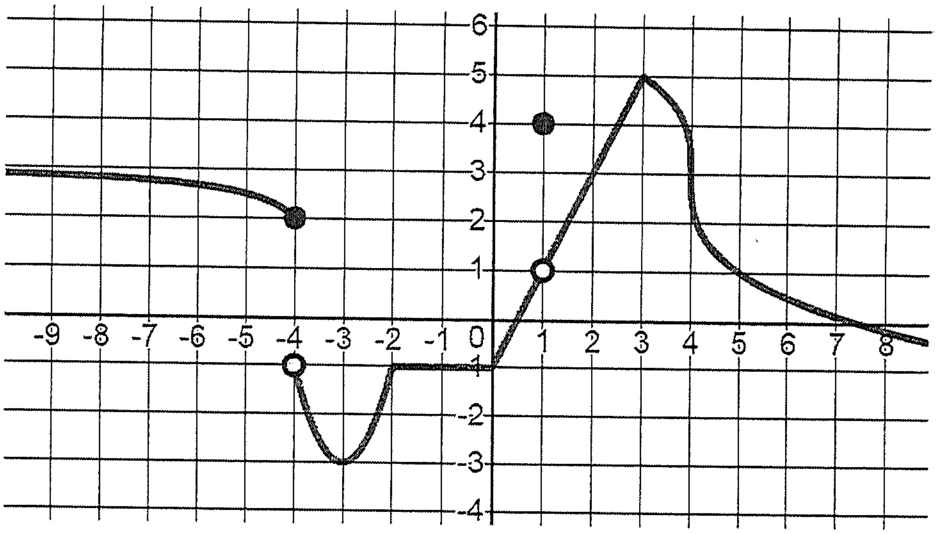
Sample Exam 1 KEY

1. Given the graph of $f(x)$, give a numerical value or state that the value does not exist. [37]



(2 points each):

$f(-4) = 2$ $f(1) = 4$

$\lim_{x \to -4^-} f(x) = 2$ $\lim_{x \to 1^-} f(x) = 1$

$\lim_{x \to -4^+} f(x) = -1$ $\lim_{x \to 1^+} f(x) = 1$

$\lim_{x \to -4} f(x) = \text{DNE}$ $\lim_{x \to 1} f(x) = 1$

$\lim_{x \to -\infty} f(x) = 3$ $f'(-3) = 0$ horiz. tan line

$m_{tan} = f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = 2$ $f'(1) = \text{DNE}$ discont $\Rightarrow$ NOT diff.

m_{tan} to right of $x=-2$ $= \lim_{x \to -2^+} f'(x) = 0$ $f'(2) = 2$ m_{tan}

(3 points each): For the graph of $f(x)$ above, answer the following questions.

Is $f(x)$ continuous at $x = 4$? Briefly explain.

yes $\lim_{x \to 4} f(x) = f(4) = 3$

Is $f(x)$ differentiable at $x = 4$? Briefly explain.

No b/c there's a vertical tangent line.

Will $f'(6)$ be positive, negative, zero, or not exist? Briefly explain.

Negative b/c f is decreasing @ $x=6$.

2. Evaluate the following limits. Give a numerical answer, if it exists. If the limit is infinite, write ∞ or $-\infty$ as appropriate. If the limit does not exist, write DNE.
Show your work and justify your answers using algebra! Do NOT use a table of values or a graph to justify your answers.

(a) $\lim_{x\to 3} \frac{2x^2-6x}{x+3} = \frac{2(3)^2-6(3)}{3+3} = \frac{2(9)-18}{6} = \frac{0}{6} = 0$ [3]

(b) $\lim_{x\to 2^+} \frac{x^2+1}{2-x}$ Type $\frac{1+}{0-}$ 2^2+1 is positive since $x>2$, $2-x$ will be negative [3]

$= -\infty$

(c) $\lim_{x\to -2} \frac{x^2-4}{x+2} = \lim_{x\to -2} \frac{\cancel{(x+2)}(x-2)}{\cancel{x+2}}$ [3]

$= \lim_{x\to -2} (x-2) = -2-2 = -4$

(d) $\lim_{x\to 1} \ln(x) - \sqrt{x} = \ln(1) - \sqrt{1}$ [3]

$= 0 - 1$

$= -1$

3. (a) Finish the definition of the derivative started below: [4]

Given a function $f(x)$, we define the **derivative** of $f(x)$ to be the limit, if it exists,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

(b) Calculate the derivative of $f(x) = 3x^2 - x$ using the definition. [8]

$$f'(x) = \lim_{h \to 0} \frac{[3(x+h)^2 - (x+h)] - [3x^2 - x]}{h}$$

$$= \lim_{h \to 0} \frac{[3(x^2+2xh+h^2) - (x+h)] - [3x^2 - x]}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{3x^2} + 6xh + 3h^2 - \cancel{x} - h - \cancel{3x^2} + \cancel{x}}{h}$$

$$= \lim_{h \to 0} \frac{6xh + 3h^2 - h}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(6x + 3h - 1)}{\cancel{h}}$$

$$= \lim_{h \to 0} (6x + 3h - 1)$$

$$= 6x + 3(0) - 1$$

$$= 6x - 1$$

4. Given that $f(x) = x^2(2x-1)$ and $f'(x) = x(6x-2)$, find the equation of the tangent line to the graph of f at $x = 2$. [4]

pt: $f(2) = 2^2(2(2)-1) = 4(3) = 12 \rightarrow (2, 12)$

slope: $f'(2) = 2(6(2)-2) = 2(10) = 20 \rightarrow m = 20$

eq'n: $y - 12 = 20(x-2)$ OR $y = 20x - 28$

5. Let $T = f(A)$ be the temperature T of the atmosphere in degrees Celsius at an altitude of A kilometers above the surface of the earth.

(a) What does $T(\overset{km}{60}) = \overset{°C}{-33}$ mean in the context of the above scenario? [4]

At 60 km above the earth, the temperature of the atmosphere is $-33°C$.

(b) What does $T'(\overset{km}{60}) = -\frac{1}{5}$ $\overset{°C/km}{\text{mean?}}$ [4]

At 60 km above the earth, the temperature will drop by 1°C for every 5 km further from the earth.

(c) What does $T^{-1}(\overset{°C}{10}) = \overset{km}{3}$ mean? [4]

The temperature of the atmosphere is 10°C at 3 km above the earth.

(d) What does $(T^{-1})'(\overset{°C}{10}) = \overset{km/°C}{4}$ mean in English? [4]

When the temperature is 10°C, it will take an increase of 4 kilometers away from the earth to raise the temperature by 1°C.

(e) At 90 km above the earth's surface, the temperature will rise by 2 degrees Celsius for every additional kilometer moved away from the earth's surface. How would you write this fact in mathematical notation? [4]

$f'(90) = 2$

6. Let $f(x) = \begin{cases} 6 - x & x < 1, \\ 3 & x = 1 \\ 3 + 2x & x > 1. \end{cases}$

(a) Use limits to determine if $f(x)$ is continuous at $x = 1$. [4]

$\lim_{x\to 1^-}(6-x) = 6-1 = 5$

$\lim_{x\to 1^+}(3+2x) = 3+2(1) = 5$

$\Rightarrow \lim_{x\to 1} f(x) = 5$

$f(1) = 3$

Since $\lim_{x\to 1} f(x) \neq f(1)$, f is NOT continuous @ $x=1$.

(b) Is $f(x)$ differentiable at $x = 1$? Explain why or why not. [4]

No, because f is NOT continuous @ $x=1$.

7. Sketch the graph of a SINGLE function $f(x)$ which satisfies ALL of the following conditions: [7]

   1. $f(2) = 3$,
   2. $\lim_{x \to 2} f(x) = -1$,
   3. $f(-3) = 1$,
   4. $\lim_{x \to -3^+} f(x) = 4$,
   5. $f'(x) > 0$ on $-\infty < x < -3$ and $2 < x < \infty$
   6. $f'(x) < 0$ on $-3 < x < 2$.