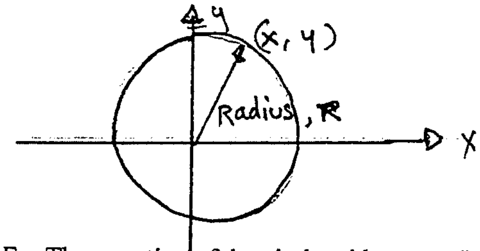
LECTURE NOTES- CHAPTER 10, SECTION 10.4(CIRCLES)

DEFINITION – A circle is the set of all points (x, y) in a plane at a distance ' R' from a given point called the center. The distance ' R' is the radius of the circle.



EQUATION OF THE CIRCLE – The equation of the circle with center (h,k) in standard form is

$$(x-h)^2+(y-k)^2=R^2$$

REMARKS – The equation of the circle can be derived from the conic equation $x^2+y^2+dx+Ey+F=0$

by completing the square. Consider the following derivation :

group similar terms $x^2+dx+y^2+Ey+F=0$

complete the square $x^2+dx+\frac{d^2}{4}-\frac{d^2}{4}+y^2+Ey+\frac{E^2}{4}-\frac{E^2}{4}+F=0$

$$\left(x+\frac{d}{2}\right)^2+\left(y+\frac{E}{2}\right)^2=\frac{d^2+E^2-4F}{4}$$

The center is at the point $\left(-\frac{d}{2}, -\frac{E}{2}\right)$ and the radius $R=\frac{1}{2}\sqrt{d^2+E^2-4F}$

Now, if $d^2+E^2-4F>0$ – circle is real

$d^2+E^2-4F<0$ – imaginary expression

$d^2+E^2-4F>0$ – Radius is zero, And equation simply represents the point $\left(-\frac{d}{2},-\frac{E}{2}\right)$

a.)

EXAMPLE – Write the equation of the circle with its center at the point (-2, 3) and radius 4 in standard form.

given: $h=-2$, $k=3$, $R=4$

using $(x-h)^2+(y-k)^2=R^2$

$$(x-(-2))^2+(y-3)^2=(4)^2$$

$$(x+2)^2+(y-3)^2=16$$

b.) write the equation of the circle with its center at the point (-2,3) And Radius 4 as a conic equation.

given: $h=-2$, $k=3$, $R=4$

using: $x^2+y^2+dx+Ey+F=0$

$h=-\frac{d}{2}$, $k=-\frac{E}{2}$, $R=\frac{1}{2}\sqrt{d^2+E^2-4F}$

substituting: $-2=-\frac{d}{2}$, $3=-\frac{E}{2}$

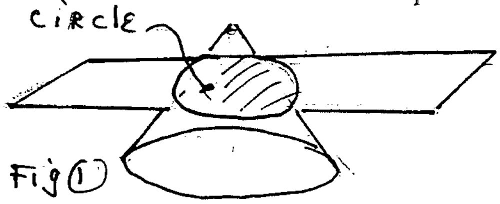
$d=4$, $E=-6$

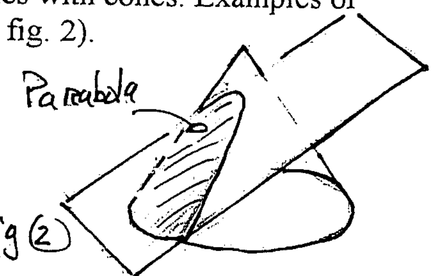
thus $x^2+y^2+4x-6y-3=0$

EXPLORING CONIC SECTIONS—CHAPTER 9

PARABOLAS

The ancient Greeks explored the intersections of planes with cones. Examples of intersections include circles and parabolas (fig. 1 and fig. 2).



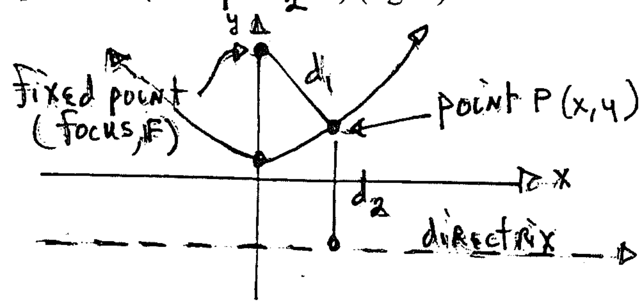


Circles and parabolas are examples of conic sections. A conic section is formed by the intersection of plane and a cone. A formal definition of a conic is also provided:

Def. The path of a point which moves so that its distance from a fixed point is in constant ratio, 'e', to its distance from a fixed line is called a conic section, or conic.

Def. The fixed point is the **focus** of the conic, the fixed line is the **directrix**, and the constant ratio, **e**, is the eccentricity.

If e = 1, the conic is a parabola ($e = d_1 / d_2 = 1$) (fig. 3)



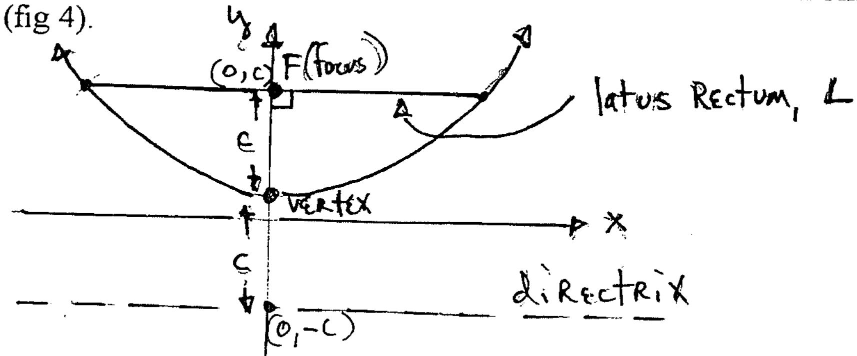
Conics are represented by the general equation $Ax^2 + Bx + Cy^2 + Dy + Exy + F = 0$.
The equation of the parabola

$$y = a(x - h)^2 + k \quad \text{or } y - k = a(x - h)^2$$

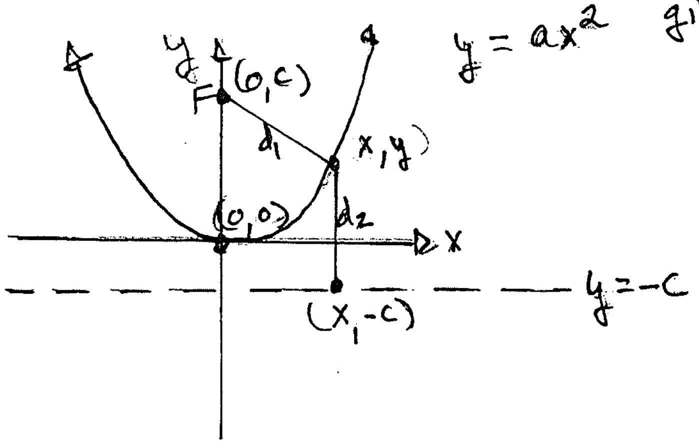
is derived from the general equation of a conic by completing the square.

Def. **Latus Rectum** is a chord through the focus and perpendicular to the axis of symmetry (fig 4). The length of the latus rectum is characterized by **L**.

Def " **c** " is the distance between the vertex and the focus. It is also the distance from the vertex to directrix. (fig 4).



Consider the equation $y - K = a(x - h)^2$, where h = 0 and k = 0. The quantity " **a**" can be expressed in terms of "**c**". The derivation is given below.



$y = ax^2$ given parabola

$d_1 = d_2$

$\Rightarrow d_1^2 = d_2^2$

$(x-0)^2 + (y-c)^2 = (x-x)^2 + (-c-y)^2$

$x^2 + \cancel{y^2} - 2cy + \cancel{c^2} = \cancel{c^2} + 2cy + \cancel{y^2}$

$x^2 = 2cy + 2cy$

$x^2 = 4cy$

$\Rightarrow y = \left(\frac{1}{4c}\right) x^2$

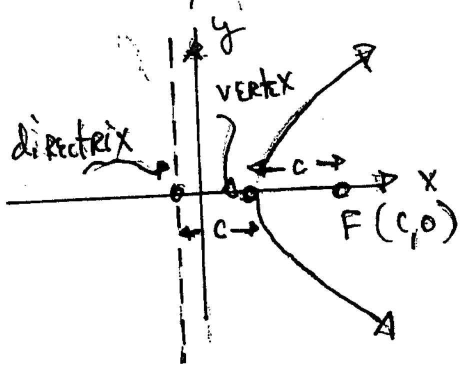
Recall, $y = ax^2$

thus $a = \frac{1}{4c}$

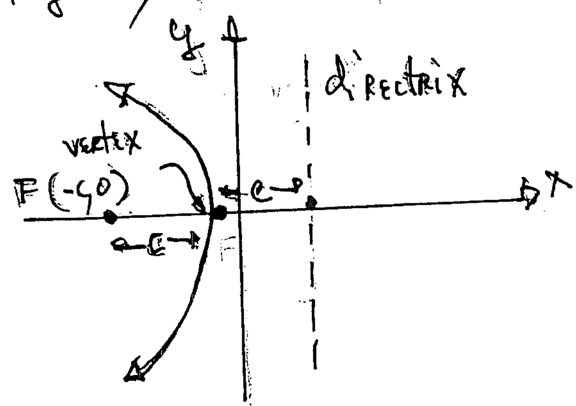
(also expressed as $|a| = \frac{1}{4c}$)

For $a > 0$, parabola opens 'up'; for $a < 0$, parabola opens 'down'

Note Parabolas can also open sideways. This case is described by the equation $x - h = a(y - k)^2$. For $a < 0$, parabola opens left
for $a > 0$, parabola opens 'right'.

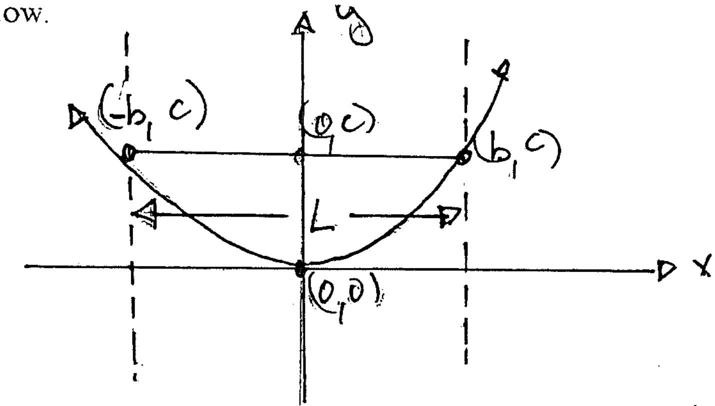


$a > 0$



Latus Rectum

The length of the latus rectum, L, can be expressed as a function of "c". The derivation is shown below.



$y = ax^2$, for the figure shown.

for $x = \pm b$, $y = c$

substituting

$$c = ab^2$$

$$\Rightarrow b^2 = \frac{c}{a}$$

$$b = \pm\sqrt{\frac{c}{a}}$$

'L' is the distance $b - (-b) = 2b$

$$\Rightarrow 2b = +2\sqrt{\frac{c}{a}}$$ (note: distance is positive!)

since $a = \frac{1}{4c}$

$$L = 2\sqrt{\frac{c}{\frac{1}{4c}}} = 2\sqrt{\frac{4c^2}{1}} = 2(2c)$$

$$L = 4c$$

IB ALGEBRA II

LECTURE NOTES – CHAPTER 10, SECTIONS 10.4 (ELLIPSE) AND 10.5 (HYPERBOLA)

DEFINITION -An ellipse is the set of all points (x,y), the sum of whose distances from two distinct fixed points (foci) is constant (see figure).

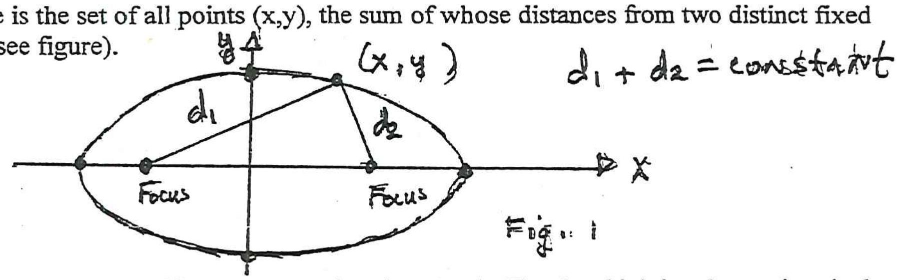


Fig. 1

The line through the foci intersects the ellipse at two points (**vertices**). The chord joining the vertices is the **major axis**, and its midpoint is the center of the ellipse. The chord perpendicular to the major axis at the center is the **minor axis** of the ellipse. The minor axis intersects the ellipse at two points (**co-vertices**) (see figures below).

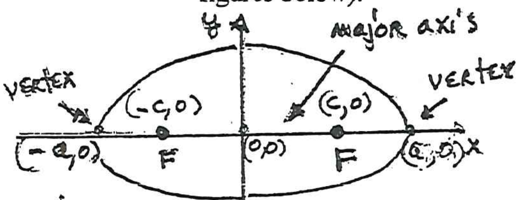


Fig 2

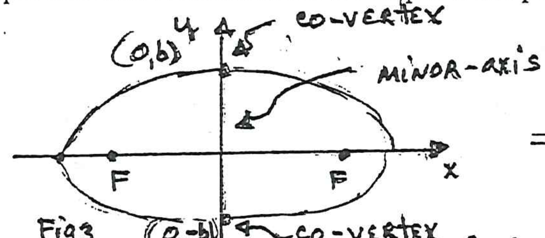


Fig 3

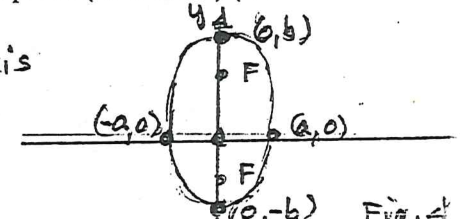


Fig. 4

REMARKS – The foci lie on the **major axis** , c units from the center, with $c^2 = a^2 - b^2$ (for a > b [fig 2]), or $c^2 = b^2 - a^2$ (for b > a [fig 4]).

EQUATION OF THE ELLIPSE – The equation of the ellipse in standard form is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \qquad \text{Equation (1)}$$

WRITING THE EQUATION OF THE ELLIPSE IN STANDARD FORM – Consider the general conic equation $Ax^2 + By^2 + Cx + Dy + E = 0$. By grouping the appropriate terms and completing the square, the equation of the ellipse in standard form may be derived. An example is given.

example – Write the following equation of the ellipse in standard form $x^2 + 4y^2 + 6x - 8y + 9 = 0$

group similar terms: $x^2 + 6x + 4y^2 - 8y + 9 = 0$

complete the square: $x^2 + 6x + 9 - 9 + 4(y^2 - 2y + 1 - 1) + 9 = 0$

$(x+3)^2 + 4(y-1)^2 - 4 = 0$

$(x+3)^2 + 4(y-1)^2 = 4$

divide then by '4'

$$\frac{(x+3)^2}{4} + \frac{(y-1)^2}{1} = 1$$ ⇒ standard form!

DERIVATION – Derive equation (1) using the ellipse shown below.

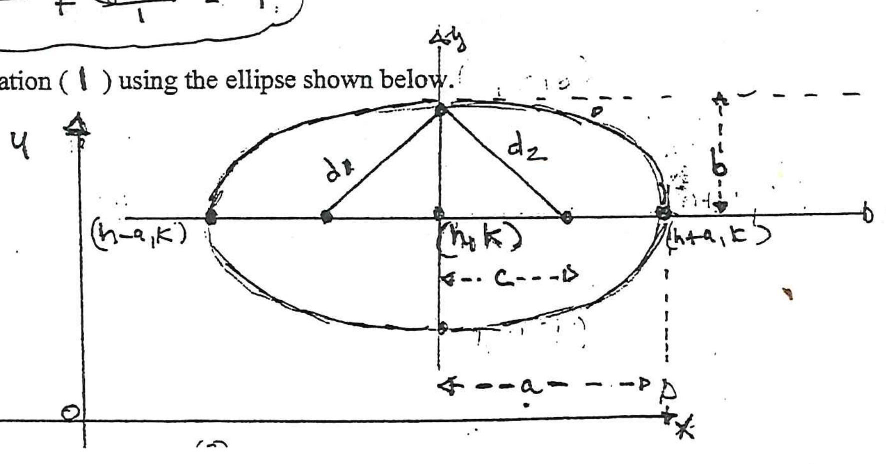


Fig. 5

Let the center of the ellipse have coordinates (h,k), vertices ($h \pm a$, k) and foci ($h \pm c$, k). Let d_1 and d_2 represent the distance from each of the foci to a point (x,y) on the ellipse. Now, for each point (x,y) of the ellipse, the average of the distances d_1 and d_2 is the given number a; that is, $\frac{1}{2}(d_1 + d_2) = a$, or $d_1 + d_2 = 2a$ (the length of the major axis). Thus

$$\sqrt{[x-(h-c)]^2+(y-k)^2} + \sqrt{[x-(h+c)]^2+(y-k)^2} = 2a$$

From figure 5 we can see that $b^2 = a^2 - c^2$, which implies that the equation of the ellipse is

$$b^2(x-h)^2 + a^2(y-k)^2 = a^2b^2$$

Dividing through by a^2b^2, we get

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

APPLICATIONS – An application involving an elliptical orbit

The moon travels about the earth in an elliptical orbit with the earth at one focus, as shown in the figure below. The major and minor axes of the orbit have lengths of 768, 806 kilometers and 767,746 kilometers respectively. Find the greatest and least distances (the apogee and perigee) from the earth's center to the moon's center.

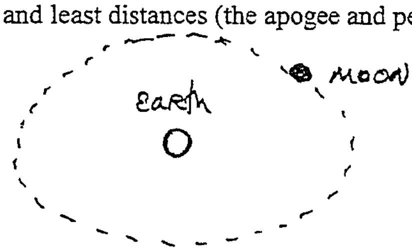


Fig. 6

Solution –

Since 2a = 768,806 and 2b = 767,746, you have a = 384,403, b = 383,873. Thus $c = \sqrt{a^2 - b^2} \approx 20{,}179$.

Therefore, the greatest distance between the center of the earth and the center of the moon is

$a + c \approx 404{,}582$ km

and the least distance is $a - c \approx 364{,}224$ km

ECCENTRICITY- One of the reasons it was difficult for early astronomers to detect that the orbits of the planets are ellipses is that the foci of the planetary orbits are relatively close to their centers, thus making the orbits nearly circular. To measure the 'ovalness' of an ellipse, we use the concept of eccentricity. The eccentricity of an ellipse is given by the ratio

To see how this ratio is used to describe the shape of an ellipse, note that since the foci of an ellipse are located along the major axis between the vertices and the center, it follows that $0 < c < a$.

For an ellipse that is nearly circular, the foci are close to the center and the ratio c/a is small (see figure below). On the other hand, for an elongated ellipse, the foci are close to the vertices, and the ratio c/a is close to 1.

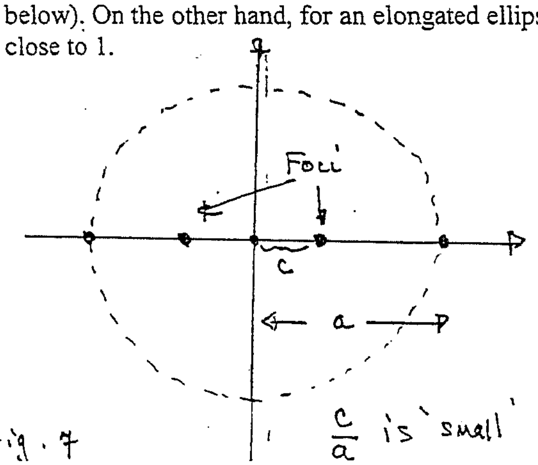


Fig. 7 $\frac{c}{a}$ is 'small'

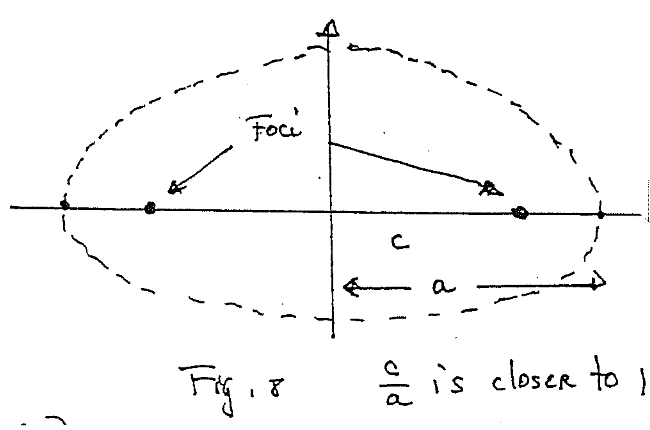


Fig. 8 $\frac{c}{a}$ is closer to 1

The orbit of the moon has an eccentricity of e = 0.0549, and the eccentricities of some planetary orbits are as follows.

Mercury: e = 0.2056
Venus: e = 0.0068
Pluto: e = 0.2481

Halley's comet, interestingly, has an elliptical orbit with an eccentricity of e = 0.97

Problems

✓ ① Given the ellipse whose equation is $4x^2+9y^2-48x+72y+144=0$, find its center, vertices, co-vertices and foci. Sketch.

② For the ellipse with equation $9x^2+16y^2=576$, find the length of the major axis, minor axis, semi-major axis, semi-minor axis, coordinates of the foci, coordinates of the co-vertices, sketch.

✓ ③ Derive the equation of the ellipse having its center at the origin, one focus at (0,3) and length of semi-major axis "5."

④ Determine the equation of the ellipse with foci $(0, \pm 4)$ and which passes through $(\frac{12}{5}, 3)$. Sketch. Give coordinates of vertices and length of major axis and semi-minor axis.

⑤ Find the equation of the ellipse having its center at the origin, major axis on the x-axis, and passing through the points (4,3) and (6,2). b.) Find the foci coords, and c) sketch.

① $4x^2-48x+9y^2+72y+144=0$

$4(x^2-12x)+9(y^2+8y)+144=0$

$\frac{1}{2}(-12)=-6$ $\quad$ $\frac{1}{2}(8)=4$

$(-6)^2=36$ $\quad$ $(4)^2=16$

$4([x^2-12x+36]-36)+9([y^2+8y+16]-16)+144=0$

$4[(x-6)^2-36]+9[(y+4)^2-16]+144=0$

$4(x-6)^2-144+9(y+4)^2-\cancel{144}+\cancel{144}=0$

$4(x-6)^2+9(y+4)^2=144$

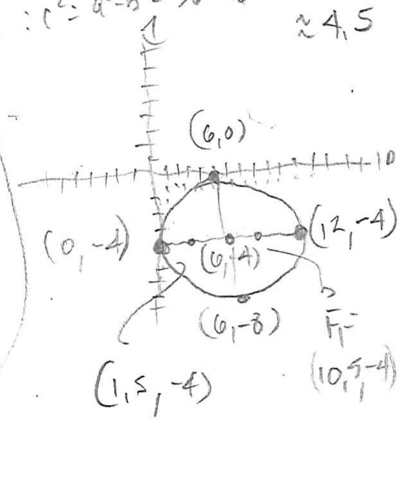
divide thru by 144

$\frac{(x-6)^2}{36}+\frac{(y+4)^2}{16}=1$

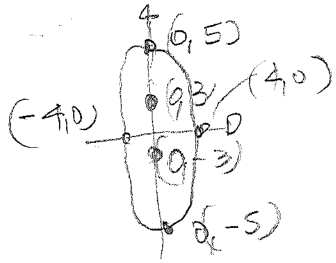
$h=6,\ k=-4$

major axis x-axis $\{a^2=36,\ a=\pm 6\}$ $\quad$ $\{b^2=16,\ b=\pm 4\}$ minor axis y-axis

Foci: $c^2=a^2-b^2=36-16=20=2\sqrt{5}$
≈ 4.5



③ . $h=0, k=0$



$b = \pm 5$

$b^2 = 25$

$c^2 = b^2 - a^2$

$9 = 25 - a^2$

$a^2 = 25 - 9 = 16$

$a = \pm 4$