8-11-2022

Kinetic and Potential Energy
Time-dependence

①

It is generally true that the force a particle experiences depends on the particle's position with respect to other bodies. This is the case, for example, with electrostatic and gravitational forces. It is also applies to the forces of elastic tension or compression. If the force is independent of velocity or time, then the differential equation for rectilinear motion is simply

$$F(x) = m\ddot{x}$$

There are several methods for solving this, including writing the acceleration as

$$\ddot{x} = \frac{d\dot{x}}{dt} = \frac{dx}{dt}\frac{d\dot{x}}{dx} = v\frac{dv}{dx}$$

(Do you recall this from earlier in the semester?)

so, the differential of motion becomes

$$F(x) = mv\frac{dv}{dx} = \frac{m}{2}\left(\frac{dv^2}{dx}\right) = \frac{dE_K}{dx}$$

The quantity $E_K = \frac{1}{2}mv^2$ is called the kinetic energy of the particle. Integrating $F(x)$, we get

$$\int F(x)\,dx = \int dE_K = \frac{1}{2}m\dot{x}^2 + \text{constant}$$

The integral $\int F(x)\,dx$ is the work done on the particle by the impressed force $F(x)$.

Given a conservative field of force, there is a function $V(x)$ such that

$$F(x) = -\frac{dV(x)}{dx}$$

The function $V(x)$ is called the potential energy. It is defined to

within an additive (arbitrary) constant. In terms of U(x), the work integral is

$$\int F(x)\,dx = \int\left(-\frac{dU}{dx}\right)dx = -U(x) + \text{constant} \qquad (2)$$

Consequently, we may write

$$E_K + U = \tfrac{1}{2}mv^2 + U(x) = \text{constant} = E,$$

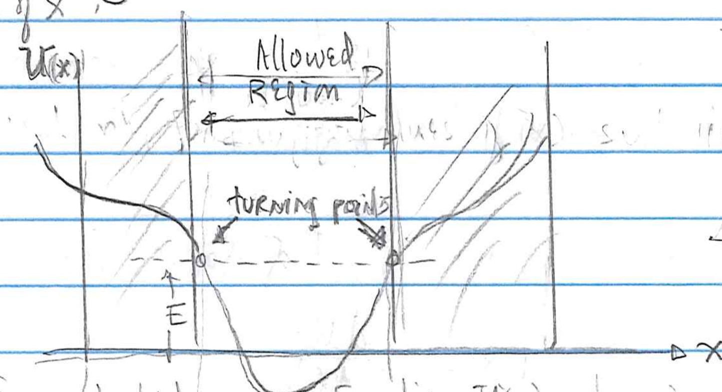
where E is the total energy. For 1-dimensional motion, if the impressed function is a function of position only, then the sum of the kinetic and potential energies remains constant throughout the motion. (Recall, this force is said to be conservative.) Nonconservative forces, for which no potential function exists, are usually of a dissipative nature, such as friction.

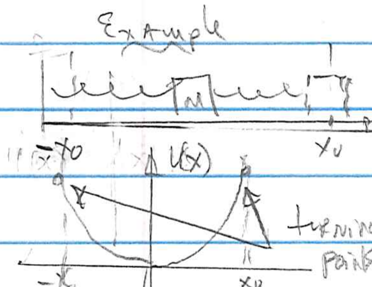
The equation of the particle can be obtained by by solving $E = \tfrac{1}{2}mv^2 + U(x)$ for v;

$$v = \pm\sqrt{\frac{2}{m}\left[E - U(x)\right]} = \frac{dx}{dt}$$

which can be expressed in integral form as

$$t = \int \frac{\pm\,dx}{\sqrt{\frac{2}{m}\left[E - U(x)\right]}},$$ thus giving "t" as a function of x.





Graph of a potential energy function U(x) showing allowed region of motion and the turning points for a given value of the total energy E

3-11-2022

turning points

Physically, the particle is confined to a region(s) for which $U(x) \le E$ is satisfied. If v goes to zero, E_k is zero, so $U(x) = E$. This means that the particle must come to rest and reverse its motion at those points for which the equality holds. These points are called the turning points of the motion.

(classwork)
Example A body is projected upward with initial speed v_i. Choosing $x=0$ (x-axis is 'upward') as the initial point of projection, 1) write an equation for E, 2) find the turning point as the particle travels upward, 3) write an equation for 't' by integrating the energy equation

The Force as a Function of Time. The Concept of Impulse

If the force acting on a particle is known explicitly as a function of time then the equation of the motion is

$$F(t) = m\frac{dv}{dt}$$

Integrating, we get

$$\int F(t)\,dt = mv(t) + C \quad , \quad C - \text{constant}$$

The integral $\int F(t)\,dt$ is called the impulse, and is equal to the momentum imparted to the particle by the force $F(t)$. The position of the particle can be found by a second integration, as follows

$$x = \int v(t)\,dt = \int \left[\int \frac{F(t')}{m}\,dt' \right] dt$$

Remark It should be noted that only in the case of the force being given as a function of t is the solution of the equation expressible as a double integral. In all other cases, the various methods of solving 2nd order differential equations must be used to find the position x as a function of t.

Classwork → 51 **Example** A block is initially at rest on a smooth horizontal surface. At time $t=0$ a constantly increasing horizontal force is applied: $F=ct$. Find the velocity and displacement as a function of time.

Variation of Gravity with Height

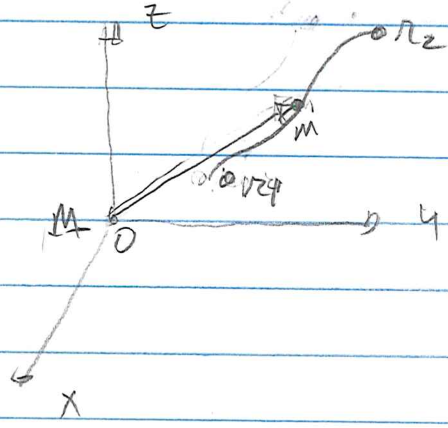
We oftentimes take 'g' to be constant. Actually, the gravitational attraction of the earth on a body above the surface falls off as the inverse square of the distance (Newton's law of gravity!). Thus, the gravitational force is

$$F = -\frac{GMm}{r^2}, \quad \begin{array}{l} G - \text{gravitational constant} \\ M - \text{mass of earth} \\ r - \text{distance from center of earth to the body} \end{array}$$

Classwork Given this force for $F(r)$, derive the potential function $U(r)$ associated with this force. In doing so, consider the work W required to move a test particle of mass m along some prescribed path in a gravitational

(classwork problem — cont'd)

Field of Another particle of mass, M. An external force $\vec{F}$ is applied to overcome the attractive gravitational force of M on the test particle m. i) Write an expression to find the work done dW for moving the test particle a distance $d\vec{r}$ along some prescribed path. ii) (then ii) using this expression, find the associated potential function $U(r)$



(recall $F = -\nabla U(r) = -\frac{dU(r)}{dr}$

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If we neglect air resistance, the differential equation for vertical motion is

$$m\ddot{r} = -\frac{GM_e m}{r^2}$$

this is clever!

$a = \frac{dv}{dt}$
$a = \frac{dv}{dx}\frac{dx}{dt}$
$= \frac{dv}{dx} \cdot v$
$= v\frac{dv}{dx}$
$= \frac{dx}{dt}\frac{\left(\frac{dx}{dt}\right)}{dx}$
$= \dot{x} \cdot \frac{d\dot{x}}{dx}$

Writing $\ddot{r} = \frac{\dot{r}\,d\dot{r}}{dr}$, we can integrate with respect to r

to get $\frac{1}{2}m\dot{r}^2 - \frac{GM_e m}{r} = E$, where E is the constant of integration, and this is the energy equation, where the sum of the potential energy $\frac{GM_e m}{r}$ and kinetic energy remain constant E

Application: Consider a projectile shot upward from the surface of the earth. The constant E is given by the initial condition

$$\frac{1}{2}mv_0^2 - \frac{GM_e m}{r_e} = E \qquad r_e - \text{radius of earth}$$

For a height x above the earth, we get $E = \frac{1}{2}mv_0^2 - \frac{GM_e m}{r_e + x}$

The speed at any height x is found as follows —

$$\frac{1}{2}m\dot{r}^2 - \frac{GM_e m}{r} = \frac{1}{2}mv_0^2 - \frac{GM_e m}{r_e + x}$$

$$\frac{1}{2}m\dot{r}^2 = \frac{1}{2}mv_0^2 + \frac{GM_e m}{r} - \frac{GM_e m}{r_e + x}$$

$$\frac{1}{2}\cancel{m}\dot{r}^2 = \frac{1}{2}\cancel{m}v_0^2 + GM_e\cancel{m}\left(\frac{1}{r} - \frac{1}{r_e + x}\right)$$

Substitute $\dot{r} = v$ and factor out a minus, we get

$$\text{so}, \quad v^2 = v_0^2 - 2GM_e\left(\frac{1}{r_e + x} - \frac{1}{r}\right)$$

At the earth's surface, $g = \frac{GM_e}{r_e^2}$

$$v^2 = v_0^2 + \underbrace{2 g r_e^2 \left(\frac{1}{r_e + x} - \frac{1}{r_e} \right)}$$

$$2g \left(\frac{r_e^2}{r_e + x} - \frac{r_e^2}{r_e} \right)$$

$$2g \left(\frac{r_e^2}{r_e + x} - r_e \right) = v_0^2 + 2g \left(\frac{r_e^2 - r_e [r_e + x]}{r_e + x} \right)$$

$$2g \left(\frac{\cancel{r_e^2} - \cancel{r_e^2} - r_e x}{r_e + x} \right)$$

$$-2gx \left(\frac{r_e}{r_e + x} \right) = -2gx \left(\frac{1}{1 + \frac{x}{r_e}} \right) = -2gx \left(1 \cdot \left[1 + \frac{x}{r_e} \right]^{-1} \right)$$

$$= -2gx \left(1 + \frac{x}{r_e} \right)^{-1}$$

so

$$v^2 = v_0^2 - 2gx \left(1 + \frac{x}{r_e} \right)^{-1}$$

if $r_e \gg x$, then $v^2 \approx v_0^2 - 2gx$

The turning point of the motion of the projectile, that is, the maximum height attained by setting $v = 0$ and solving for x, we get

$$v_0^2 = 2gx \left(1 + \frac{x}{r_e} \right)^{-1} = 2gx \left(\frac{1}{1 + \frac{x}{r_e}} \right)$$

$$\left(1 + \frac{x}{r_e} \right) v_0^2 = 2gx$$

$$\frac{x v_0^2}{r_e} + v_0^2 = 2gx$$

$$-v_0^2 \quad -2gx \quad -v_0^2$$

$$x \left(\frac{v_0^2}{r_e} - 2g \right) = -v_0^2$$

$$h = x_{MAX} = \frac{-v_0^2}{\left(\frac{v_0^2}{r_e} - 2g \right)} = \frac{v_0^2 (r_e)}{v_0^2 - 2g r_e}$$

$$h = \frac{v_0^2 r_e}{2g r_e - v_0^2} = \frac{v_0^2}{2g}\left(\frac{r_e}{r_e - \frac{v_0^2}{2g}}\right)\frac{\left(\frac{1}{r_e}\right)}{\left(\frac{1}{r_e}\right)}$$

$$h = \frac{v_0^2}{2g}\left(\frac{1}{1 - \frac{v_0^2}{2g r_e}}\right) = \frac{v_0^2}{2g}\left(1 - \frac{v_0^2}{2g r_e}\right)^{-1}$$

[Remark: for $2g r_e >> v_0^2$, then $h \approx \frac{v_0^2}{2g}$]

Can you determine the escape speed of a mass using this equation for 'h'? If so, find it. (Discuss your reasoning)