From page 9

cont'd - Orbits in an Inverse-Square Field

Recall $$r = r_0\left(\frac{1+e}{1+e\cos\theta}\right)$$

r is the value of r_0 for $\theta = 0$. The value of r for $\theta = \pi$ is given by

$$r_1 = r_0\frac{1+e}{1-e}$$

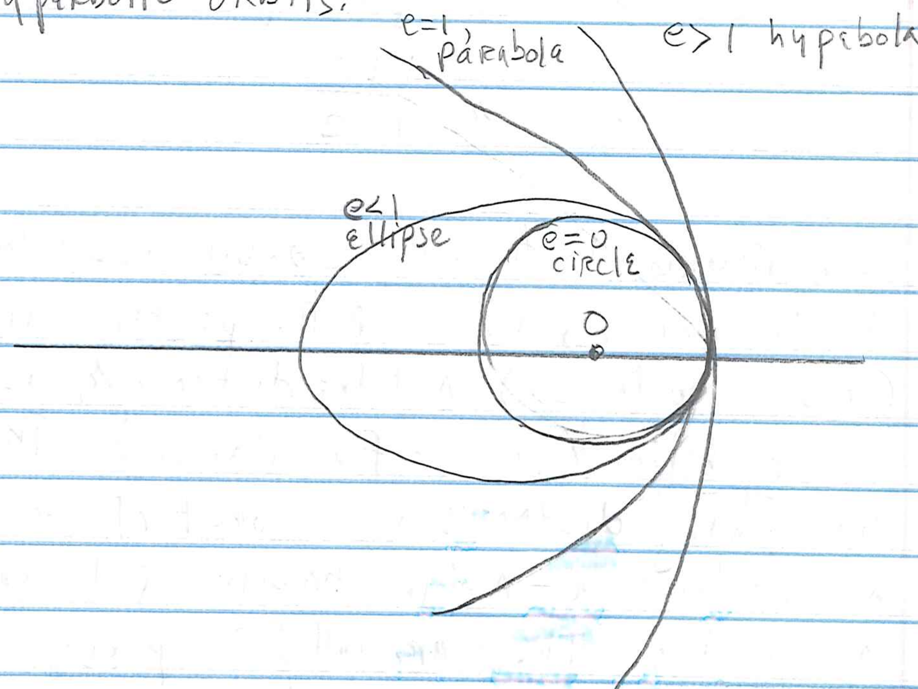
In reference to the elliptic orbits about the sun, the distance r_0 is called the perihelion distance (closest to the sun) and the distance r_1 is called the aphelion (farthest from the sun). The corresponding distances for the orbit of the moon around the earth — and the orbits of the earth's artificial satellites — are called the perigee and the apogee distances, respectively.

The orbital eccentricities of the planets are small (see table)

TABLE

Planet	*Semimajor Axis in Astronomical Units*	*Period in Years*	*Eccentricity*
Mercury	0.387	0.241	0.206
Venus	0.723	0.615	0.007
Earth	1.000	1.000	0.017
Mars	1.524	1.881	0.093
Jupiter	5.203	11.86	0.048
Saturn	9.539	29.46	0.056
Uranus	19.19	84.02	0.047
Neptune	30.06	164.8	0.009
Pluto	39.46	247.7	0.249

For example, in the case of the earth's orbit $e = 0.017$, $r_0 = 91{,}000{,}000$ miles and $r_1 = 95{,}000{,}000$ miles. On the other hand, comets generally have large orbital eccentricities (highly elongated orbits). Halley's comet, for instance, has an orbital eccentricity of 0.967 with a perihelion distance of 55,000,000 miles, while at aphelion it is beyond the orbit of Neptune. Many comets (the nonrecurring type) have parabolic or hyperbolic orbits.



The family of central conics is shown in the figure

Orbital Parameters from Initial Conditions

From the equation $r_0 = \frac{mh^2}{k(1+e)}$, we find the eccentricity can be expressed as

$$e = \frac{mh^2}{kr_0} - 1$$

Let v_0 be the speed of the particle at $\theta = 0$. Then, from the definition of the constant h we have

$$h = r^2\dot{\theta} = r_0^2\dot{\theta}_0 = r_0 v_0$$

The eccentricity is then given by

$$e = \frac{m r_0 v_0^2}{k} - 1$$

For a circular orbit $(e=0)$, $k = m r_0 v_0^2$ or

$$\frac{k}{r_0^2} = \frac{m v_0^2}{r_0}$$

Let's let $\frac{k}{m r_0} = v_e^2$, so that if $v_0 = v_e$,

the orbit is a circle. The equation for eccentricity e can then be written as

$$e = \left(\frac{v_0}{v_e}\right)^2 - 1$$

And the equation of the orbit becomes

$$r = r_0 \frac{\left(\frac{v_0}{v_e}\right)^2}{1 + \left[\left(\frac{v_0}{v_e}\right)^2 - 1\right]\cos\theta}$$

The value of r_1 is given by $\theta = \pi$, thus

$$r_1 = r_0 \frac{\left(\frac{v_0}{v_e}\right)^2}{2 - (v_0/v_e)^2}$$

Example A rocket satellite travels around the earth in a circular orbit of radius r_0. A sudden blast of the rocket motor increases the speed by 10%. Find the equation of the new orbit, and compute the apogee distance. Let v_e be the speed of the circular orbit, and let v_0 be the new

initial speed, that is $v_0 = 1.1\, v_e$

1) write the equation of the new orbit,

2) the apogee distance

3) sketch the orbits, including their radii; 4) indicate perigee and apogee.

Solutions (classwork)

Orbital energies in the Inverse square Field

Since the potential energy function $U(r)$ for an inverse-square force field is given by

$$U(r) = -\frac{k}{r} = -ku$$

the energy equation of the orbit $E = \frac{1}{2} m h^2 \left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] + U(u)$

now reads

$$\frac{1}{2} m h^2 \left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] - ku = E$$

or upon separating variables

$$d\theta = \left(\frac{2E}{mh^2} + \frac{2ku}{mh^2} - u^2\right)^{-1/2} du$$

Integrating, we get

$$\theta = \sin^{-1}\left[\frac{mh^2u - k}{(k^2 + 2Emh^2)^{1/2}}\right] + \theta_0$$

where θ_0 is a constant of integration. If let $\theta_0 = -(\pi/2)$ and solve for u, we obtain

these are lower case 'u'

$$u = \frac{K}{mh^2}\left[1 + (1 + 2Emh^2 K^2 K^{-2})^{1/2} \cos\theta\right]$$

or

$$r = \frac{mh^2 K^{-1}}{1 + (1 + 2Emh^2 K^{-2})^{1/2}\cos\theta}$$

This is the polar equation of the orbit! If we compare it with the equation $r = r_0 \frac{1+e}{1+e\cos\theta}$ and $e = \frac{Amh^2}{K}$, we see that the eccentricity is given by

$$e = (1 + 2mEh^2K^{-2})^{1/2}$$

The above expression for the eccentricity allows us to classify the orbits according to the total energy E, as follows

$E < 0$ $e < 1$: closed orbits (ellipse or circle)
$E = 0$ $e = 1$: parabolic orbit
$E > 0$ $e > 1$: hyperbolic orbit

Since $E = E_K + U$ and is constant, the closed orbits are those for which $E_K < |U|$, and the open orbits are those for which $E_K \geq |U|$.

Application A comet is observed to have a speed v_0 when it is a distance r_0 from the sun, and its direction of motion makes angle φ with the radius vector from the sun. Find the eccentricity of the comet's orbit.

Solution In the Sun's gravitational field $k = GMm$, where M is the mass of the sun and m is the mass of the body. The total energy is then given by

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r} = \frac{1}{2}mv_0^2 - \frac{GMm}{r_0} = \text{constant}$$

and the orbit will be elliptic, parabolic, or hyperbolic, according to whether E is negative, zero or positive. Accordingly, if v_0^2 is less than, equal to, or greater than $\frac{2GM}{r_0}$, the orbit will be an ellipse, a parabola, or a hyperbola, respectively.

Now, $h = |\vec{r} \times \vec{v}| = r_0 v_0 \sin\varphi$

The eccentricity from the equation

$$e = (1 + 2Emh^2k^{-2})^{1/2}$$ has the value

$$e = \left[1 + \left(v_0^2 - \frac{2GM}{r_0}\right)\frac{r_0^2 v_0^2 \sin^2\varphi}{G^2M^2}\right]^{1/2}$$

Next installment: 1) Limits of radial motion

2) Periodic Time of orbital motion