Classical Mechanics
1-8-2022

'VARiations on a Theme'

WARMUP
Review problem
$\nabla \varphi$ (gradient)

①

Thus far, we have used the Euler-Lagrange equation in conjunction with the variables x & y. We have seen how to use the formulation $ds = \sqrt{dx^2 + dy^2}$ to find or minimize the distance between points. We also investigated the brachistochrone problem where we wrote the equation of a curve that minimized the time for a mass traveling between 2 points.

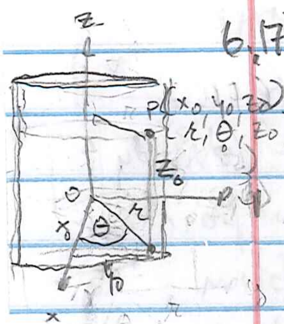
Today, we will 'branch out' and use 2 examples wherein we write the E-L equation for coordinates other than x & y.

Generalized coordinates review -
Recall, we generalized the formulation of the E-L equation notation to represent various coordinate systems, and used the variables q_i and $\dot{q}_i$, which could, for example, be used to represent polar coordinates.

In the examples that follow, we will in fact be using other coordinates besides x, y, and z.

Definition: Geodesic is commonly a curve that represents the shortest path between 2 points on a surface (or more generally, a Riemannian manifold)

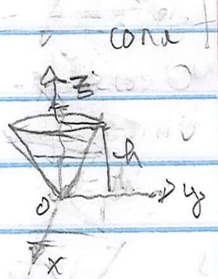
e.g., solid sphere



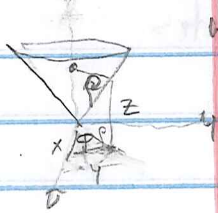
6.7) Find the geodesics on a cone whose equation in cylindrical coordinates is $z = \lambda\rho$. Let the required curve have the form $\varphi = \varphi(\rho)$,

dependent variable — independent variable

Recall, for cylindrical coordinates, $x = r\cos\theta$, $y = r\sin\theta$, $z = z$,



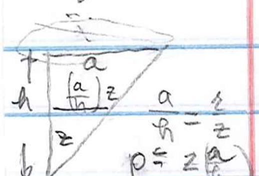
So, $ds = \sqrt{dx^2 + dy^2 + dz^2} = \sqrt{d\rho^2 + \rho^2 d\varphi^2 + dz^2}$



where $z = \lambda\rho$

so $\frac{dz}{d\rho} = \lambda \Rightarrow dz = \lambda\, d\rho$ and $dz^2 = \lambda^2 d\rho^2$

substituting



$$ds = \sqrt{d\rho^2 + \rho^2 d\varphi^2 + \lambda^2 d\rho^2} = \sqrt{d\rho^2(1+\lambda^2) + \rho^2 d\varphi^2}$$

get used to doing this

Given that $\varphi = \varphi(\rho)$, then $d\varphi = \left(\frac{d\varphi}{d\rho}\right) d\rho = \dot{\varphi}\, d\rho$

$\Rightarrow d\varphi^2 = \dot{\varphi}^2 d\rho^2$

Now, we have everything in terms of $d\rho$, so we obtain

$$ds = \sqrt{(1+\lambda^2)\, d\rho^2 + \rho^2 \dot{\varphi}^2 d\rho^2} = \sqrt{(1+\lambda^2) + \rho^2 \dot{\varphi}^2}\; d\rho$$

$$L = \int ds = \int f\, d\rho, \quad \text{where } f = f(\varphi, \varphi', t) = \left[1 + \lambda^2 + \rho^2 \varphi'^2\right]^{1/2}$$

then $\frac{\partial f}{\partial \varphi} - \frac{d}{d\rho}\frac{\partial f}{\partial \dot{\varphi}} = 0$

dependent variable — independent variable

③

$$\frac{\partial f}{\partial \varphi} = 0 \Rightarrow \frac{\partial f}{\partial \dot\varphi} = \text{constant} = K \leftarrow \text{"dummy variable"}$$

$$\frac{\partial f}{\partial \dot\varphi} = \frac{1}{2}\left(1+\lambda^2+\rho^2\dot\varphi^2\right)^{-\frac{1}{2}}\left(2\dot\varphi\rho^2\right) = K$$

$$\frac{\rho^2\dot\varphi}{(1+\lambda^2+\rho^2\dot\varphi^2)^{1/2}} = K$$; it is convenient to square both sides of the equation:

$$\frac{\rho^4\dot\varphi^2}{1+\lambda^2+\rho^2\dot\varphi^2} = K^2$$

simplifying —

$$\rho^4\dot\varphi^2 = K^2\left(1+\lambda^2+\rho^2\dot\varphi^2\right)$$

$$\rho^4\dot\varphi^2 = K^2 + K^2\lambda^2 + K^2\rho^2\dot\varphi^2$$

$$\dot\varphi^2\left(\rho^4 - K^2\rho^2\right) = K^2\left(1+\lambda^2\right)$$

$$\dot\varphi^2\rho^2\left(\rho^2-K^2\right) = K^2\left(1+\lambda^2\right)$$

$$\dot\varphi^2\rho^2 = \frac{K^2(1+\lambda^2)}{\rho^2-K^2}$$

$$\dot\varphi = \left[\frac{K(1+\lambda^2)}{\rho^2-K^2}\right]^{\frac{1}{2}}$$

$$\frac{d\varphi}{d\rho} = \rho\left[\frac{K(1+\lambda^2)}{\rho^2-K^2}\right]^{\frac{1}{2}}, \quad \text{let } \alpha = \left[K(1+\lambda^2)\right]^{\frac{1}{2}}$$

$$\int_{\varphi_0}^{\varphi} d\varphi = \int \frac{\alpha}{\left[\rho^2\rho^2-K^2\right]^{\frac{1}{2}}}\,d\rho, \quad \varphi-\varphi_0 = \frac{\alpha}{K}\sec^{-1}\left|\frac{\rho}{K}\right| = \frac{K(1+\lambda^2)^{\frac{1}{2}}}{K}\sec^{-1}\left(\frac{\rho}{K}\right) =$$

$$= (1+\lambda^2)^{\frac{1}{2}}\sec^{-1}\left(\frac{\rho}{K}\right)$$

$$\frac{\varphi-\varphi_0}{1+\lambda^2} = \sec^{-1}\left(\frac{\rho}{K}\right) \Rightarrow \sec\left[\frac{\varphi-\varphi_0}{1+\lambda^2}\right] = \frac{\rho}{K}, \quad \rho = \frac{K}{\cos\left(\frac{\varphi-\varphi_0}{1+\lambda^2}\right)}$$

in the limit as $\lambda \to 0$, the cone approaches the plane $z = 0$ and the geodesic approaches the curve

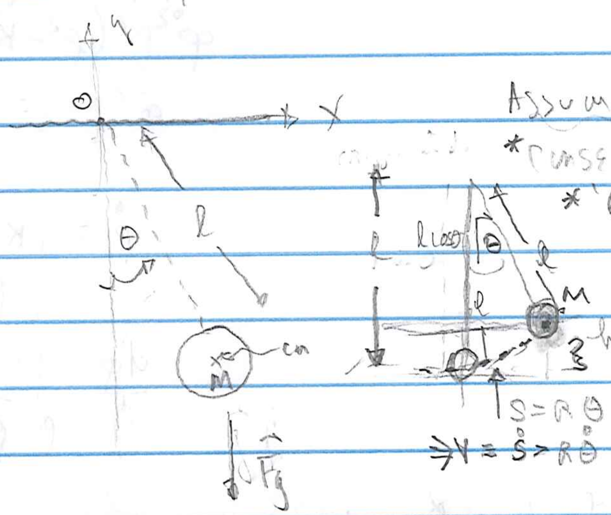
$$\rho = \frac{k}{\cos(\varphi - \varphi_0)}$$ (which can be shown to be a straight line perpendicular to the direction $\varphi - \varphi_0$ a distance k from the origin)

This is difficult to see, at the moment

Reminder - we will address the Lagrangian more formally in a future lecture.

The next example involves use of the Lagrangian in a very easy application using the simple pendulum. From this scenario, we see how the E-L equation is incorporated and used to help us determine the equations of motion.

$$\underset{\text{Lagrangian}}{\mathcal{L}} = \underset{\text{kinetic energy}}{E_K} - \underset{\text{potential energy}}{U}$$ (sometimes expressed as $\mathcal{L} = T - V$)



Assumptions
* conservative system
* 'l' is constant

To obtain the equations of motion, we write the Lagrangian formulation using generalized coordinates

$$\frac{\partial L}{\partial q_i} = \left(\frac{d}{dt}\right)\frac{\partial L}{\partial \dot{q}_i}$$

we will see this represents force

this represents momentum

The generalized coordinates are $q_i = \theta$, $\dot{q}_i = \dot{\theta}$

Let's use the geometry from the problem to write expressions for kinetic and potential energy

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$$E_K = \frac{1}{2} m v^2 = \frac{1}{2} m (l \dot{\theta})^2 = \frac{1}{2} m l^2 \dot{\theta}^2$$

Note that the kinetic energy, E_K, depends on $\dot{\theta}$

The potential energy is $U = mgh$, where $h = l - l\cos\theta = l(1-\cos\theta)$, so the potential energy, U, is a function of the coordinate θ

Now, we may write the Lagrangian, $\mathcal{L}$

$$\mathcal{L} = E_K - U = \frac{1}{2} m l^2 \dot{\theta}^2 - \underbrace{mgl(1-\cos\theta)}_{mgl + mgl\cos\theta}$$

$\mathcal{L}$ is a functional: $\mathcal{L} = \mathcal{L}[\theta, \dot{\theta}, t]$; taking partial derivatives, we get

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \cancel{2}\left(\frac{1}{\cancel{2}}\right) m l^2 \dot{\theta} \qquad \frac{\partial \mathcal{L}}{\partial \theta} = -mgl\sin\theta \ (\neq 0), \text{ so } \frac{\partial \mathcal{L}}{\partial \dot{\theta}} \text{ is \underline{not a constant}},$$

thus $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = m l^2 \ddot{\theta}$, and

$$\frac{\partial \mathcal{L}}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = -mgl\sin\theta - ml^2\ddot{\theta} = 0$$

$$\Rightarrow \quad ml^2\ddot{\theta} + mgl\sin\theta = 0$$

simplifying, $\ddot{\theta} + \left(\frac{g}{l}\right)\sin\theta = 0$

Warm up (review) problem for next class <Wednesday>

- solve this differential equation
- write expressions for the frequency, angular frequency, and the period