4-15-2021
Classical mechanics

The Brachistochrone Problem

This is a very well-known application of the Euler-Lagrange equation and the calculus of variations.

REMARK
"βραχύς (στο)" is from the Greek, meaning "shortest", and "χρόνος" translates shortest time

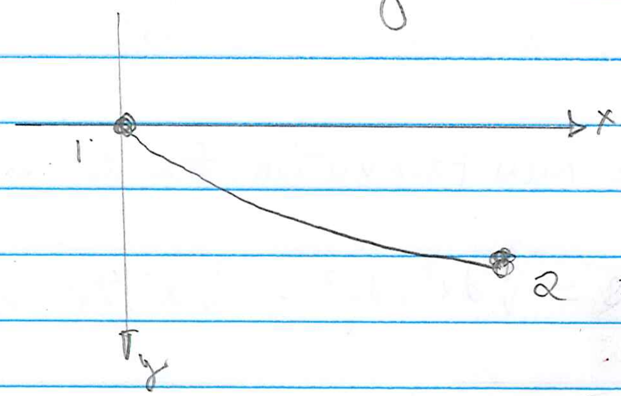

Fig (1)

Remark
Let P_1 be at $y=0$ and P_2 be at y_2

Consider a roller coaster track, as shown in Fig (1). What shape must the track have so that a cart released from point 1 will reach point 2 in the shortest possible time? It is assumed that the track is frictionless.

This is similar to our introductory problem for the E-L equation, where we minimized length, that is, $L = \int ds = \int \sqrt{1+y'^2}\, dx$ where the independent variable is x ; in this next application, we will minimize t and the independent variable will be y. Recall, using conservation of energy,

$$\Delta KE = -\Delta U$$

$$\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = -(mgh_f - mgh_i)$$

$$v_i = 0,\ h_i = 0$$

hence, $\frac{1}{2}mv_f^2 = -(mgh_f)$

$$\Rightarrow \quad v = v_f = \sqrt{2gy}$$

To incorporate time and length, recall

$v = \frac{ds}{dt}$, so that $dt = \frac{ds}{v}$, where $v = \sqrt{2gy}$

Because this gives v as a function of y, we will take 'y' as our independent variable, so we will alter our distance formula to reflect this change

As such, $x = x(y)$, so that $\frac{dx}{dy} = \left(\frac{dx}{dy}\right) dy$

$dx = x' \, dy$, $dx^2 = (x')^2 dy^2$

And our new formulation for ds becomes

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{(x')^2 dy^2 + dy^2} = \sqrt{dy^2(1+(x')^2)} =$$

$$= \left(\sqrt{1+(x')^2}\right) dy$$

Recall, $t = \int_1^2 \frac{ds}{v} = \int_0^{y_2} \frac{\left(\sqrt{1+(x')^2}\right)}{\sqrt{2gy}} dy$,

where point 1 is at $y = 0$ and point 2 is at y_2

Rearranging a bit, we get

$$t = \frac{1}{\sqrt{2g}} \int_0^{y_2} \frac{\sqrt{1+(x')^2}}{\sqrt{y}} dy$$, where

$$f[x, x', y] = \frac{\sqrt{1+(x')^2}}{\sqrt{y}}$$

Now, we are able to use the E-L equation to minimize time, (where instead of $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$, we now have

$$\frac{\partial F}{\partial x} - \frac{d}{dy}\left(\frac{\partial F}{\partial x'}\right) = 0$$ We may rewrite

this as $$\frac{\partial F}{\partial x} = \frac{d}{dy}\left(\frac{\partial F}{\partial x'}\right)$$

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Note that $f[x, x', y] = \frac{\sqrt{(x')^2+1}}{\sqrt{y}}$ is independent of x, so

$$\frac{\partial f}{\partial x} = 0 \quad \Rightarrow \quad \frac{\partial f}{\partial x'} = \text{constant}$$

$$\frac{\partial f}{\partial x'} = \frac{1}{2\sqrt{y}}\left(x'^2+1\right)^{1/2-1}\left[2x'\right] = \lambda$$

$$\frac{1}{\sqrt{y}}\frac{x'}{\sqrt{(x')^2+1}} = \lambda$$

squaring both side –

$$\frac{1}{y}\left(\frac{(x')^2}{(x')^2+1}\right) = \lambda^2$$

$$\frac{(x')^2}{(x')^2+1} = \lambda^2 y$$

$$(x')^2 = \lambda^2 y\left((x')^2+1\right) = (x'^2)\lambda^2 y + \lambda^2 y$$

$$(x')^2 - (x')^2\lambda^2 y = \lambda^2 y$$

$$(x')^2(1-\lambda^2 y) = \lambda^2 y$$

$$(x')^2 = \frac{\lambda^2 y}{1-\lambda^2 y}$$

$$x' = \frac{\lambda\sqrt{y}}{\sqrt{1-\lambda^2 y}} \qquad \frac{\lambda\sqrt{y}}{\sqrt{\lambda^2\left(\frac{1}{\lambda^2}-y\right)}^{1/2}} = \frac{\cancel{\lambda}\sqrt{y}}{\cancel{\lambda}\left(\frac{1}{\lambda^2}-y\right)^{1/2}}$$

you may recognize this from calculus or math tools where $dx = \sqrt{\frac{y}{D-y}}\,dy$ is the differential equation of an inverted cycloid generated by a circle of

(4)

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Brachistochrone Problem

(5)

Diameter $D = 2r$, and whose equation in parametric form is $x = r(\varphi - \sin\varphi)$

And $y = r(1-\cos\varphi)$

φ is the angle through which a rolling circle forming the cycloid has rotated

Hence, $x = \int\sqrt{\frac{y}{D-y}}\,dy = \sqrt{\frac{r(1-\cos\varphi)}{2r - [1-\cos\varphi]}}\,dy$

where $\frac{dy}{d\varphi} = d\left([r-(1-\cos\varphi)]\right) = r\sin\varphi$,

or $dy = r\sin\varphi\, d\varphi$

Lets simplify some, $x = \sqrt{\frac{r(1-\cos\varphi)}{2r-[1-\cos\varphi]}}\; r\sin\varphi\, d\varphi =$

$x = \sqrt{\frac{r(1-\cos\varphi)}{2r - r(1-\cos\varphi)}}\left(r\sin\varphi\, d\varphi\right) = \sqrt{\frac{r(1-\cos\varphi)}{2r - r + r\cos\varphi}}\; r\sin\varphi\, d\varphi$

$x = \sqrt{\frac{r(1-\cos\varphi)}{r + r\cos\varphi}}\; r\sin\varphi\, d\varphi = \sqrt{\frac{\cancel{r}(1-\cos\varphi)}{\cancel{r}(1+\cos\varphi)}}\; r\sin\varphi\, d\varphi$

$x = \sqrt{\frac{1-\cos\varphi}{1+\cos\varphi}}\; r\sin\varphi\, d\varphi$

let $z = -\cos\varphi$

so that $\frac{dz}{d\varphi} = \sin\varphi$, or $dz = \sin\varphi\, d\varphi$

$pz+q$
$az+b$
$a=-1,\ b=1$
$p=1,\ q=1$

This gives us, upon substitution

$$\left[\sqrt{\frac{1+z}{1-z}}\right] r\,dz$$, integrating this

bad boy yields

$$= r\left\{\frac{\sqrt{(az+b)(pz+q)}}{a} + \frac{aq-bp}{2a}\int \frac{dz}{\sqrt{(az+b)(pz+q)}}\right\}$$

$$= r\left\{\frac{\sqrt{(-z+1)(z+1)}}{-1} + \frac{-1-1}{2(-1)}\int \frac{dz}{\sqrt{(-z+1)(z+1)}}\right\}$$

$$= r\left\{\frac{\sqrt{1-z^2}}{-1} + \cancel{\left(\frac{-2}{-2}\right)}\int \frac{dz}{\sqrt{1-z^2}}\right\}$$

since $z=-\cos\varphi$, upon substitution, we obtain:

$$r\left\{\frac{\sqrt{1-(-\cos\varphi)^2}}{-1} + \int \frac{\sin\varphi\,d\varphi}{\sqrt{1-(-\cos\varphi)^2}}\right\}$$

$$= r\left\{\frac{\sqrt{\sin^2\varphi}}{-1} + \int_0^{\varphi}\left(\frac{\sin\varphi}{\sin\varphi}\right)d\varphi\right\} = -r\sin\varphi + r\varphi,\ \text{or}$$

$x = r(\varphi - \sin\varphi)$ BAM!