Optimization WS KEY

open intervals: 1st deriv test

closed & bdd: endpt/CN table

1. Find the maximum of $f = x^2y$, where x = number 1 and y = number 2. constraint: $x+y=8$ so $y=x-8$

$f = x^2(8-x) = 8x^2 - x^3$ Domain: $0 \le x \le 8$

$f' = 16x - 3x^2 = 0$

$= x(16-3x) = 0$

$x = 0$

$16-3x=0$

$16=3x$

$\boxed{x = 16/3}$

so $y = 8 - 16/3 = \boxed{\frac{8}{3}}$

	x	f
endpoint →	0	0
endpoint →	8	0
CN →	16/3	75.85 ← global max

The two numbers that will produce the biggest value of x^2y are $x = \frac{16}{3}$ and $y = \frac{8}{3}$.

2 a) 35 people: $(60 \cdot 35) = 2100$ 36 ppl: $59 \cdot 36 = 2124$ 37 ppl: $58 \cdot 37 = 2146$

b) x = number of people over 35 people

price per person: $60 - x$

$R(x) = (35+x)(60-x)$ b/c = quantity · price

Check formula: 36 people: $R(36) = \$2124$ ✓

37 people: $R(37) = \$2146$ ✓

c) Maximized when $R'(x) = 0$

$R(x) = 2100 + 25x - x^2$ Domain: $0 \le x \le 15$

$R'(x) = -2x + 25 = 0$

$2x = 25$

$x = 25/2$ → cannot have half people so check 12 and 13

x	$R(x)$
0	2100
15	2250
12	2256
13	2256

12 and 13: both global max.

Revenue is maximized at a total of 47 or 48 people. (47 = 35+12, 48 = 35+13)

d $R(13) = (35+13)(60-13) = \$2256$ ∴ the maximum revenue.

3 w = width l = length

constraint:

$P = l + 2w$

$500 = l + 2w$

$l = 500 - 2w$

objective function: $A = lw$

$A = (500-2w)(w) = 500w - 2w^2$

$A' = 500 - 4w$

$0 = 500 - 4w$

$4w = 500$

$\boxed{w = 125}$ $l = 250$

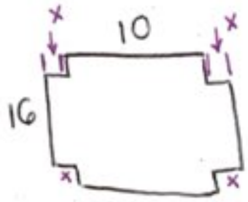
need $l > 0$ → Domain: $0 < w < \frac{500}{2} = 250$

1st derivative test: A' + + | − − at 125

125 ~~global~~ local max + since only CP, global max

max area = $(250)(125)$

$= 31250$



4 $l = 10-2x$ $\quad$ Volume $= lwh$

$w = 16-2x$

height $= x$

$V = (10-2x)(x)(16-2x)$ $\quad$ Domain: $0 < x < 5$ ← $10-2x=0$ is lowest the length can be

$V = (10x - 2x^2)(16-2x)$

$= 160x - 20x^2 - 32x^2 + 4x^3$

$V' = 160 - 40x - 64x + 12x^2 = 0$

$160 - 104x + 12x^2 = 0$

$4(3x^2 - 26x + 40) = 0$ $\quad (3x-20)(x-2) = 0$

$x-2=0 \Rightarrow x = 2$ → maximum

$3x-20=0 \Rightarrow x = \frac{20}{3}$ ← outside of domain

V' ++ $\big|_2$ − − − $\quad$ 1st deriv. test

2 → global max

Largest box

$V = (10-2(2))(2)(16-(2\cdot 2)) = 144\text{ in}^3$ $\quad l = 10-2(2) = 6$ $\quad w = 16-2(2) = 12$

The dimensions of the largest box are 2in x 6in x 12in with a max volume of 144 in^3.

5. 

Square base.

$V = w^2h$ ← because $l = w$

$100 = 8w + 4h$ (8 pieces of wire of length w + 4 pieces of length h make the frame)

$100 - 8w = 4h$

$25 - 2w = h$

$V = w^2(25-2w) = 25w^2 - 2w^3$ $\quad$ Domain: $0 < w < \frac{100}{8} = 12.5$ → if $h = 0$, this is max that w can be

$V' = 50w - 6w^2 = 0$

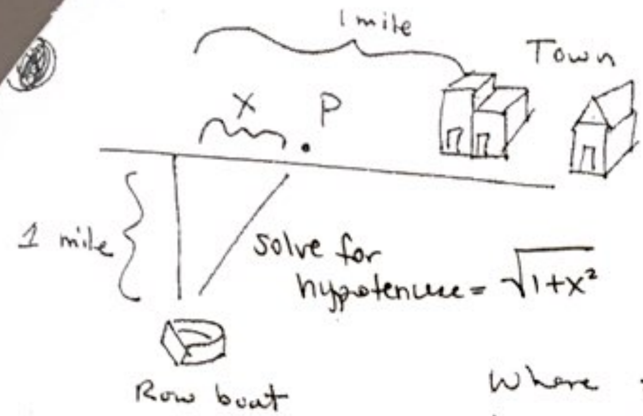
$w(50-6w) = 0$ $\quad V'$ ++ $\big|_{\frac{25}{3}}$ − − − max

$w = 0$ ← not in domain $\quad 50 - 6w = 0$

$w = \frac{-50}{-6} = \frac{25}{3}$

$w = \frac{25}{3}$, $h = 25 - 2(\frac{25}{3}) = \frac{25}{3}$.

The dimensions of the box with the largest volume are $\frac{25}{3}$cm x $\frac{25}{3}$cm x $\frac{25}{3}$cm.



The man can row @ 3 mph and can walk @ 5 mph.

Where should he land the row boat (where is point P) so that the total time is minimized?

Note velocity = $\frac{\text{distance}}{\text{time}}$ which means time = $\frac{\text{distance}}{\text{velocity}}$.

Let x represent the distance from the point on the shore closest to the man and point p.

$$T = \text{Total time} = \text{time rowing} + \text{time walking}$$

$$= \frac{\sqrt{1+x^2}}{3} + \frac{(1-x)}{5}.$$

domain is $0 \le x \le 1$ because need $0 \le 1-x$ (it's a distance) and $1-x \le 1$ (max 1 mile to town)

$$T' = \frac{1}{6}(1+x^2)^{-1/2}(2x) + \left(-\frac{1}{5}\right).$$

$$0 = \frac{1}{3}\frac{x}{\sqrt{1+x^2}} - \frac{1}{5}$$

$$\frac{3}{5} = \frac{x}{\sqrt{1+x^2}} \underset{\text{square both sides}}{\Longrightarrow} \frac{9}{25} = \frac{x^2}{1+x^2}$$, now cross-multiply

$$1 + x^2 = \frac{25}{9}x^2$$

$$1 = \frac{16}{9}x^2$$

$$\frac{9}{16} = x^2$$

$$\frac{3}{4} = x$$

He should row to a point <u>1/4</u> of a mile <u>from</u> town to minimize the total time.

	x	Time = T
endpt →	0	.53
endpt →	1	.47
cp →	3/4	.467 ← global min

7 X = length of a side of **square** end of the package
W = length of the package
$V = x^2 l$
girth + length = $4x + l = 108$
$l = 108 - 4x$ Domain $0 < x < \frac{108}{4}$
$V(x) = x^2(108 - 4x) = 108x^2 - 4x^3$
$V'(x) = 216x - 12x^2 = 12x(18 - x)$
$x > 0$ $x = 18$ V' ++ -- ← global max (at 18)
$108 = (4 \cdot 18) + l$ $l = 36$
max volume = 11664
The dimensions producing the largest volume are 18in x 18in x 36in with a volume of $11,664 \text{ in}^3$.

8 r = radius $V = \pi r^2 h$ ← constraint
h = height $\frac{355}{\pi r^2} = h$

S = area of base + area of side ← objective function
$= 2\pi r^2 + 2\pi r h$ → circumference · height
top and bottom
$= 2\pi r^2 + 2\pi r\left(\frac{355}{\pi r^2}\right)$

$= 2\pi r^2 + \frac{710}{r}$ Domain $0 < r < \infty$

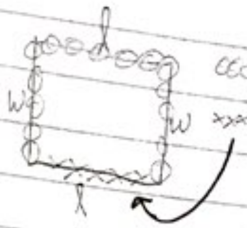
$S' = 4\pi r - \frac{710}{r^2} = 0$

$r^2\left(4\pi r - \frac{710}{r^2}\right) = 0 \cdot r^2$

$4\pi r^3 - 710 = 0$
$4\pi r^3 = 710$
$r^3 = 710 / 4\pi$
1st deriv. test
$r = \left(\frac{710}{4\pi}\right)^{1/3} \approx 3.84.$ S' -- ++ (384 min.) $h = \frac{355}{\pi(3.84^2)} \approx 7.67$

The dimensions with the least surface area are when the radius of the can is 3.84 cm and the height is 7.67 cm.

9.



ooo = shrubs @ $25/ft.
xxx = fence @ $10/ft.

objective function $C = 25w + 25l + 25w + 10l = 50w + 35l$

constraint
$A = lw = 3000$ $\quad l = \frac{3000}{w}$

$C = 50w + 35\left(\frac{3000}{w}\right) = 50w + 105000\,w^{-1}$

$C' = 50 - 105000w^{-2} = 50 - \frac{105000}{w^2} = 0$ $\quad$ Domain: $0 < w < \infty$

$\frac{-105000}{w^2} = -50 \Rightarrow -105000 = -50w^2 \Rightarrow w^2 = \frac{105000}{50} \Rightarrow w = \sqrt{\frac{105000}{50}}$

≈ 45.83

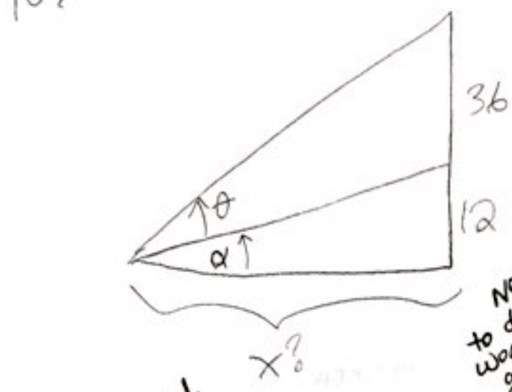
C' $- - -$ | $+ + +$
45.83
min

$l = \frac{3000}{45.83} = 65.47$

$C = 50(45.83) + 35(65.47) = 4582.58$

The lowest cost to build this garden is $4,582.58.

10.



Maximize θ

$\tan(\alpha+\theta) = \frac{48}{x}$ $\qquad 0 < x < \infty$

Need to do work to get to objective function

$\alpha + \theta = \arctan\left(\frac{48}{x}\right)$ $\qquad$ b/c $\tan\alpha = \frac{12}{x}$ $\Rightarrow \alpha = \arctan\left(\frac{12}{x}\right)$

$\theta = \arctan\left(\frac{48}{x}\right) - \alpha$

objective function $\rightarrow$ $\theta = \arctan\left(\frac{48}{x}\right) - \arctan\left(\frac{12}{x}\right)$

$\theta' = \frac{1}{1+\left(\frac{48}{x}\right)^2}\cdot\left(\frac{-48}{x^2}\right) - \frac{1}{1+\left(\frac{12}{x}\right)^2}\cdot\left(\frac{-12}{x^2}\right)$

$= \frac{x^2}{x^2+48^2}\cdot\left(\frac{-48}{x^2}\right) - \frac{x^2}{x^2+12^2}\left(\frac{-12}{x^2}\right)$

$0 = \frac{-48}{x^2+48^2} + \frac{12}{x^2+12^2}$

$\frac{48}{x^2+48^2} = \frac{12}{x^2+12^2}$

$48(x^2+12^2) = 12(x^2+48^2)$

$48x^2 + 12^2(48) = 12x^2 + 12(48^2)$

$36x^2 = 12(48^2) - 12^2(48)$

$x = \sqrt{\frac{12(48^2)-12^2(48)}{36}} = 24$

1st deriv. test

$+ + \quad | \quad - - -$

24

max

You should stand 24 in. away to maximize angle θ.