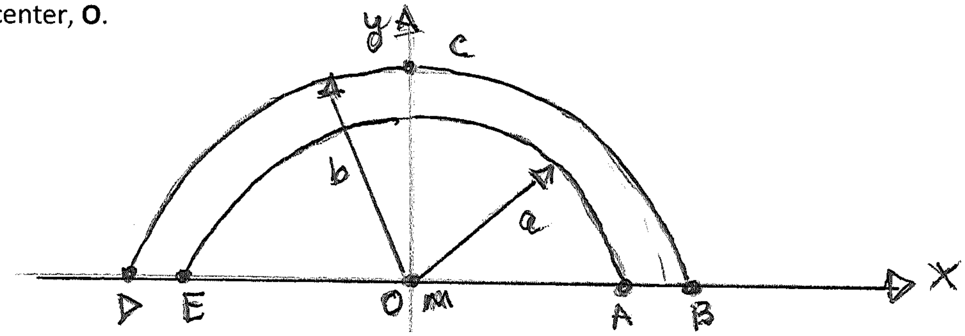
EXAM **<u>CLASSICAL MECHANICS</u>** **4-19-2022**

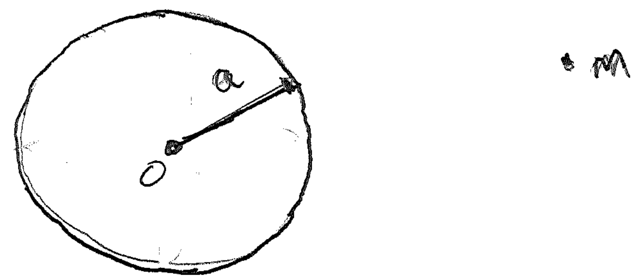
DUE NO LATER THAN APRIL 30, 2020

1. SELECT 5 OF THE PROBLEMS YOU HAVE BEEN HANDED TO WORK COMPLETELY AND CORRECTLY. ALL PROBLEMS ARE WEIGHTED EQUALLY.
2. SOLUTIONS MUST BE **HIGHLIGHTED**.
3. ALL STEPS IN SOVING PROBLEMS MUST BE INCLUDED WITHOUT 'GAPS' OR 'SHORTCUTS'. THE LOGIC/THOUGHT IN THE DEVELOPMENT OF SOLUTIONS MUST BE COHERENT AND FLOW SMOOTHLY. NO CREDIT WILL BE GIVEN FOR SOLUTIONS THAT 'MAGICALLY' APPEAR OR MAKE NO SENSE WITHIN THE CONTEXT OF THEIR DEVELOPMENT.
4. PLEASE KEEP VERBIAGE TO A MINIMUM – OBSERVE 'ECONOMY OF EXPRESSION.
5. WRITE YOUR NAME ON THE FRONT OF THE EXAM AND INITIAL EACH PAGE. MAKE SURE THAT ALL PAGES ARE STAPLED TOGETHER SECURELY.

1. A uniform plate has its boundary consisting of 2 concentric half-circles of inner and outer radii **a** and **b**, respectively, as shown in the figure. Find the force of attraction on a mass **m** located at the center, **O**.



2. Find the force of attraction of thin spherical shell of radius **a** on a particle P of mass **m** at a distance **r** > **a** from its center.



3. Prove that the path of a planet around the sun is an ellipse with the sun at one focus. USE VECTOR METHODS.

4. Prove that the speed **v** of a particle moving in an elliptical path in an inverse square field is given by

$$\mathbf{v}^2 = \frac{k}{m}\left(\frac{2}{r} - \frac{1}{a}\right)$$

where **a** is the semi-major axis

5. A man-made satellite revolves about the earth at height **H** above the surface. Determine a.) the orbital speed b.) orbital period so that a person in the satellite will be in as state of weightlessness.

6. A particle of mass m moves in the xy-plane, acted on by a linear restoring force $\mathbf{F} = -k\mathbf{r} = -k(x, y)$, where k is a positive constant. Determine and sketch the trajectory if the initial conditions at $t = 0$ are

$$\mathbf{r}_o = (x_o, 0) \quad \text{and} \quad \mathbf{v}_o = (0, v_o).$$

Label the axis on the trajectory; include the points x_0, $-x_0$, $\frac{v_o}{\omega}$, $\frac{-v_o}{\omega}$

7. A one-dimensional force $\mathbf{F} = -kx\hat{\mathbf{x}}$, where k is a constant, acts on a particle of mass m.

(a) Calculate the potential energy $V(x)$ of the particle.
(b) Sketch the possible graphs of $V(x)$. Use these graphs and conservation of energy to discuss the possible motions of the particle.

On your graphs, draw horizontal lines to denote the constant energy, label V(x), the graph where k>0, the graph where k<0, identify turning points, and forbidden regions. Use the points x_1 and x_2 to denote the initial and final points, and to help identify forbidden regions (since a particle with energy E cannot enter them).

8. Prove that in plane polar coordinates (r, θ) the velocity and acceleration vectors are given by

$$\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} \tag{1}$$

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}}. \tag{2}$$

9. *Particle in a central potential.* **Lagrangian Mechanics**

A particle of mass m moves in R^3 under a central force

$$F(r) = -\frac{\mathrm{d}V}{\mathrm{d}r},$$

in spherical coordinates, so

$$(x, y, z) = (r\cos(\phi)\sin(\theta), r\sin(\phi)\sin(\theta), r\cos(\theta)).$$

Find the Lagrangian from first principles, in terms of (r, θ, ϕ) and their time derivatives.

Hence

(a) show that h, defined by $h = mr^2\dot{\phi}\sin^2(\theta)$ is a constant of the motion.
(b) derive the other two equations of motion.

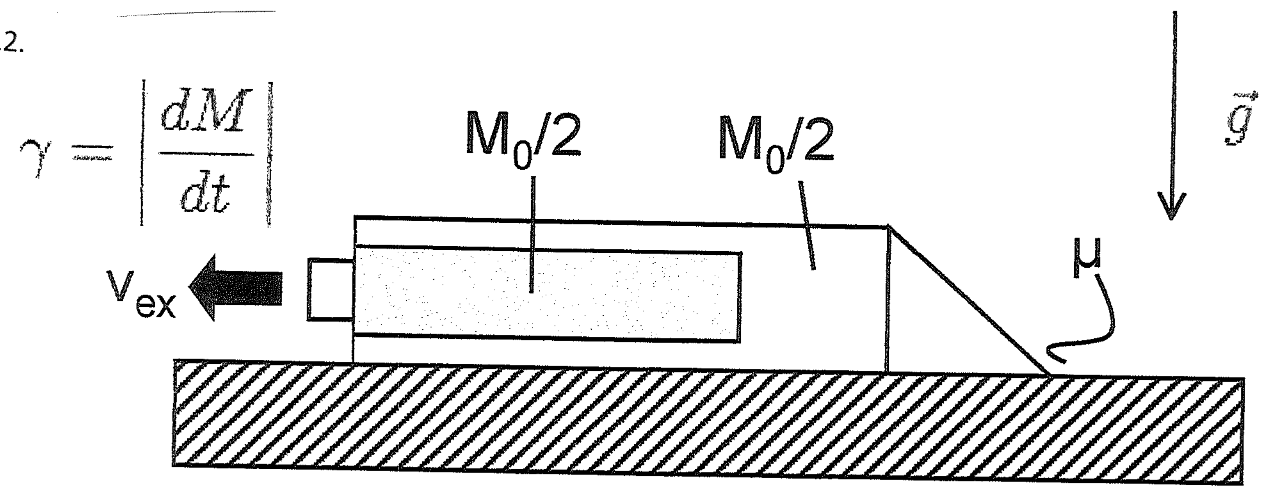
10. A particle of mass m is subject to a one-dimensional restoring force $F_r = -kx$ (k is a positive constant), a frictional force proportional to the velocity $F_f = -\alpha v$ (α is a positive constant), and a harmonic driving force $F_d = F_0 \cos \omega t$ (F_0 and ω are constants). Determine the position $x(t)$.

11. If F is the time-dependent force $F = A - Bt$, where A and B are positive constants, determine the velocity $v(t)$ and the trajectory $x(t)$ in terms of A, B, m, v_0, and x_0. Sketch the graphs of $F(t)$, $v(t)$ and $x(t)$ versus t for $v_0 = 0$ and $x_0 > 0$.

Be sure to label the point $\frac{A}{B}$ on the graph of F vs t, the points $\frac{A}{B}$ and $2\frac{A}{B}$ on the graph of v vs t, and the points $\frac{A}{B}$ and $2\frac{A}{B}$ on the graph of x vs t.

Problem Tabletop Rocket

12.



A rocket of total mass M_0, half of which is fuel, starts at rest on a long horizontal table. The coefficient of friction between the rocket and table surfaces is μ. At time t = 0, the rocket is ignited, ejecting fuel out at a constant rate $\gamma = |dM/dt|$ with velocity $\mathbf{v}_{ex}$ relative to the rocket. Constant gravitational acceleration g acts downward.

(a) What condition must be met for the rocket to start moving at t = 0?

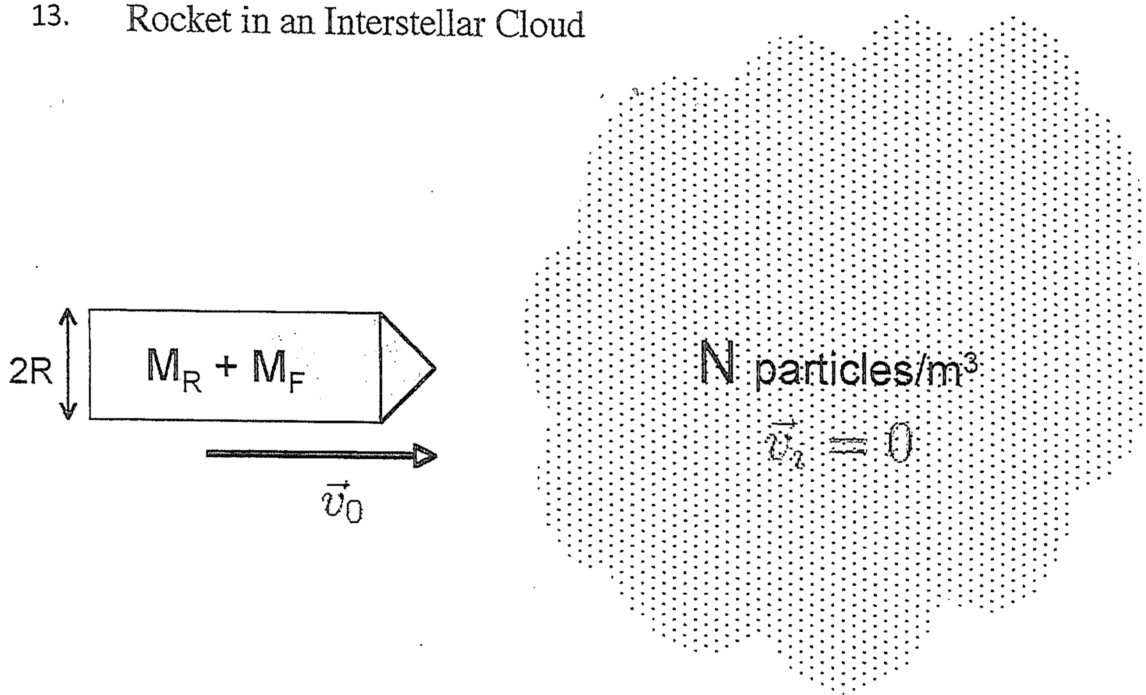
(b) Assuming that the rocket satisfies this requirement, what is the maximum speed V_{MAX} achieved by the rocket?

(c) How far does the rocket go after it runs out of fuel? You can express your answer in terms of V_{MAX}.

(d) BONUS How far does the rocket travel in total? For this you will need to make use of the following integral:

$$\int \ln u\, du = u \ln u - u$$

13. Rocket in an Interstellar Cloud



A cylindrical rocket of diameter 2R, mass M_R and containing fuel of mass M_F is coasting through empty space at velocity v_0. At some point the rocket enters a uniform cloud of interstellar particles with number density N (e.g., particles/m^3), with each particle having mass m ($\ll M_R$) and initially at rest. To compensate for the dissipative force of the particles colliding with the rocket, the rocket engines emit fuel at a rate $dm/dt = \gamma$ at a constant velocity u with respect to the rocket. Ignore gravitational effects between the rocket and cloud particles.

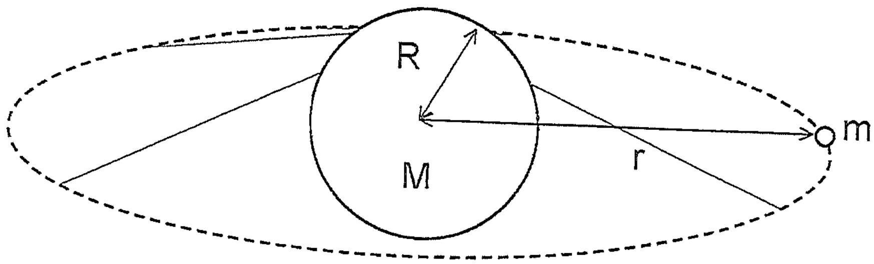
(a) Assuming that the dissipative force from the cloud particles takes the form $F = -Av^2$, where A is a constant, derive the equation of motion of the rocket ($F = ma$) through the cloud as it is firing its engines.

(b) What must the rocket's thrust be to maintain a constant velocity v_0?

(c) If the rocket suddenly runs out of fuel, what is its velocity as a function of time after this point?

(d) BONUS Assuming that each cloud particle bounces off the rocket elastically, and collisions happen very frequently (i.e., collisions are continuous), prove that the dissipative force is proportional to v^2, and determine the constant A. Assume that the front nose-cone of the rocket has an opening angle of 90°.

.1 4 Planet Orbit



A small planet of mass m is in a circular orbit of radius r around a star of mass M and radius R in otherwise empty space (assume M >> m so the star is stationary).

(a) Determine the potential energy U(r), the kinetic energy K(r) and the total mechanical energy E(r) of the planet in terms of G, M and r assuming $U \to 0$ as $r \to \infty$.

(b) Determine the minimum amount of mechanical energy that must be added to the planet to cause it to escape from the star (i.e., $r \to \infty$). By what factor must the speed of the planet be increased to cause it to escape?

(c) Now assume that the planet in subject to a viscous force of the form

$$\vec{F} = -Amv^2\hat{v}$$

where A is a constant and $\hat{v}$ is the direction of motion. Compute the loss of mechanical energy in one orbital period in terms of G, M, r and A. Assume that this loss is small enough that neither the orbital radius nor speed of the planet changes appreciably in one orbit.

(d) BONUS Building from (c), compute the change in radius of the planet in one orbital period due to the viscous force and the corresponding radial velocity based on the assumptions above, in terms of G, M, r and A. Does the planet fall into the star or away from it?