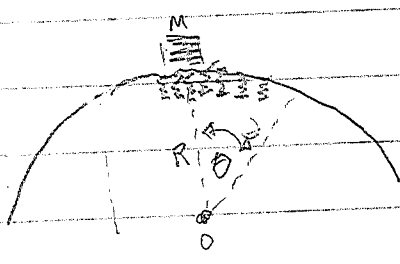
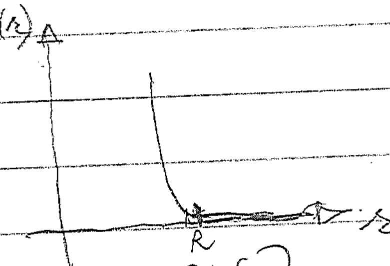
4-19-2022
Classical Mechanics

Lecture Notes / Example Problem - The Lagrangian "exotic" application

Part II - The Constraining Force

Let's say that we want to find the constraining normal force that the hemisphere exerts on the particle. To do this, let's consider an approach where we write things in terms of both r and θ. Something to consider, in the real world 'r' isn't exactly constrained to be R, because in the real world, the particle actually 'sinks in' to the hemisphere a little bit. This may seem a bit silly, but it's an important point - the particle pushes and sinks inward a tiny distance until the hemisphere gets squashed enough to push back with the appropriate amount of force to keep the particle from sinking in any more (just imagine that the hemisphere is made up of lots of little springs with large spring constants fig(1)) The particle is therefore subject to a very steep potential potential arising from the hemisphere's force (figure (2)) $U(r)$





fig(2)

The true Lagrangian for the system is thus

$$\mathcal{L} = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - mgr\cos\theta - U(r)$$

(The $\dot{r}^2$ term in the E_k will turn out to be insignificant)

The equations of motion from varying θ and r are therefore

$$mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} = mgr\sin\theta$$

$$m\ddot{r} = mr\dot{\theta}^2 - mg\cos\theta - U'(r)$$

Having written down the equations of motion, we will now apply the constraint condition that $r = R$. This implies that $\dot{r} = \ddot{r} = 0$ (of course r isn't truly equal to R, but any differences are negligible from this point forward).

The equation $mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} = mgr\sin\theta$ reproduces the equation $\ddot{\theta} = \left(\frac{g}{R}\right)\sin\theta$ from part I, while the equation $m\ddot{r} = mr\dot{\theta}^2 - mg\cos\theta - U'(r)$ yields

$$-U'(r) = -\left.\frac{dU}{dr}\right|_{r=R} = mg\cos\theta - mR\dot{\theta}^2$$

But, $F_N \equiv -\frac{dU}{dr}$ is the constraint force applied in the r direction, which is precisely the force we are looking for! The normal force of constraint is therefore $F_N(\theta, \dot{\theta}) = mg\cos\theta - mR\dot{\theta}^2$

This is equivalent to the radial $F = ma$ equation, $mg\cos\theta - F_N = mR\dot{\theta}^2$ (which is certainly a quicker way to find the normal force in the present problem). This result is valid only if $F_N(\theta, \dot{\theta}) > 0$. If the normal force becomes zero, then this means that the particle has left the sphere, in which case r no longer equals R.

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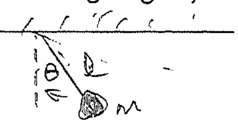
The Lagrangian, Constrained Motion, Degrees of Freedom, Applications….

Constrained Motion – When a particle is restricted geometrically in the sense that it must stay on a definite surface or curve, the motion is said to be constrained. Examples of constrained motion include a simple pendulum, a piece of ice sliding in a bowl, or a bead sliding on a wire.

Oftentimes, with constrained motion, writing the Lagrangian and subsequent equations of motion can be less of a chore as there are fewer variables needed to describe the motion. For example, when we wrote the Lagrangian for a simple pendulum, instead of writing out a force diagram, breaking up forces into their components, and using trig to write a differential equation describing the motion, we identified a single variable, θ, that we used to write the Lagrangian, and from there, we backed out the equations of motion.

Fig. 1



Today, we will examine: 1.) the simple case of a particle sliding along a hemisphere, also described with a single variable, then extend our application to determine the normal force (described as a force of constraint) acting on the particle, 2.) a mass sliding along a wire in the shape of a cycloid, 3.) a mass sliding down a moving incline, which involves the use of 2 variables to describe the motion, and time permitting, we will kick up the complexity just a bit to examine the motion of a bead on a rotating wire in the shape of a parabola (if we don't finish this last one in class or don't get to it, if you have time, you should try solving it over the weekend. It's problem 7.41 on page 288 of your text).

Before proceeding, let's revisit the simple parabola and recall that we used a single variable, θ, to express the equations of motion using the Lagrangian. Below, you will find pictured a double pendulum with masses attached to an inextensible string, then the same configuration, only one of the strings is a rubber band that is not rigid, but can stretch.

Fig 2

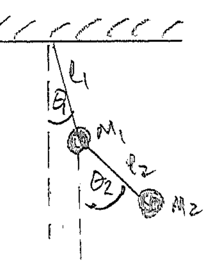
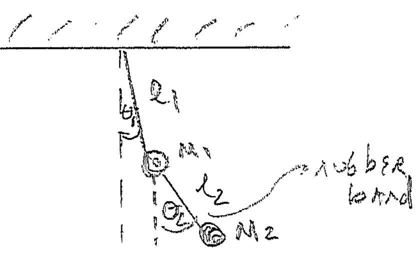


Fig 3

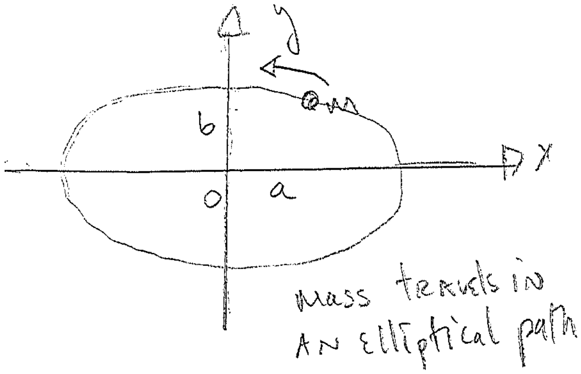


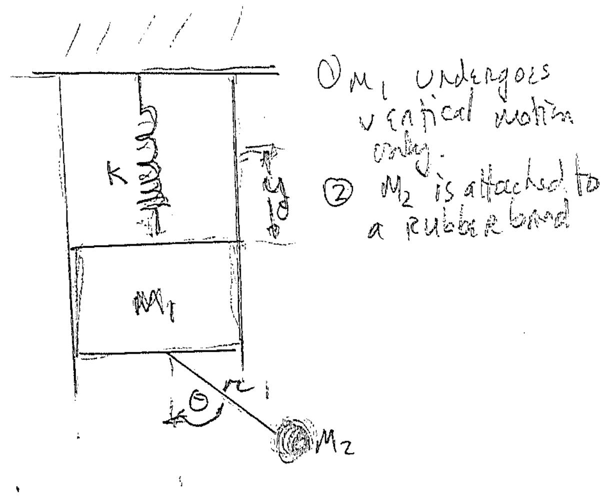
Instead of using a single coordinate to describe the motion as we did with the simple pendulum, the double pendulum having the 2 inextensible strings requires two coordinates to describe the motion, θ_1 and θ_2. What do you think about the string with the rubber band? We'll need yet a third measurement to describe the length, l_2, of the stretching rubber band.

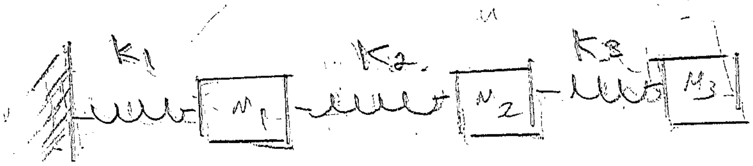
Remark: A coordinate q_k can be an angle or a distance

DEFINITION – The number of **degrees of freedom** refers to the number of coordinates that can be independently varied in a small displacement- that is, the number of directions in which a system can move from any initial configuration.

Just for practice, consider some of the sketches below and determine the number of degrees of freedom you think are associated with each one.







When the number of degrees of freedom of an N-particle system in 3 dimensions is less than 3N, the system is said to be constrained. In two dimensions, the corresponding number is 2N.

DEFINITION - A **holonomic** system has n degrees of freedom and can be described by n generalized coordinates, $q_1, q_2, \ldots, q_n$. In addition to specifying the configuration of a system, each coordinate in a holonomic system can vary independently of the others. By way of contrast, there are non-holonomic systems where the coordinates cannot vary independently. Here, the number of degrees of freedom is less than the minimum number of coordinates needed to specify the configuration. Because of their complexity, Taylor tells that he is limiting his applications to holonomic systems only.

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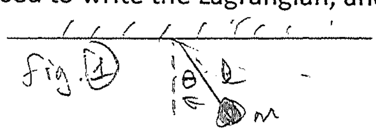

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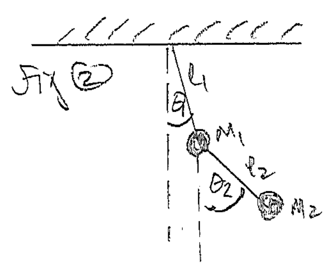

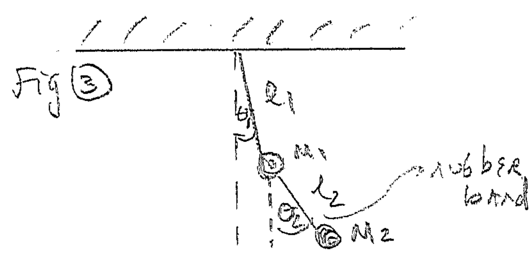

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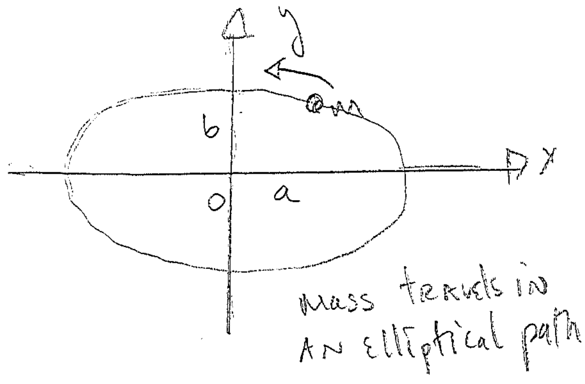

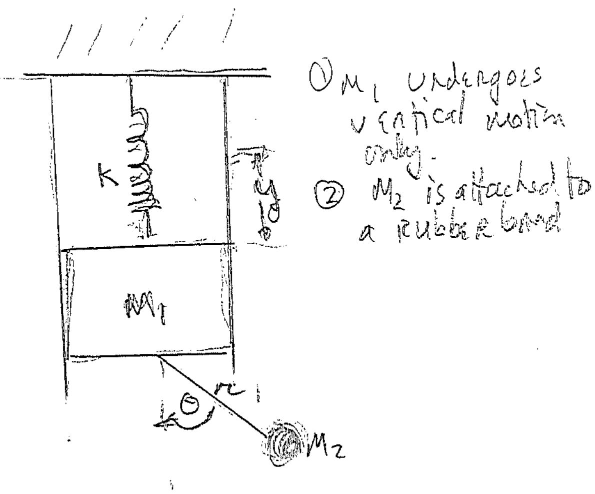

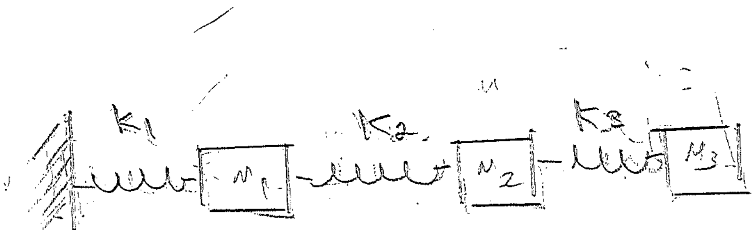

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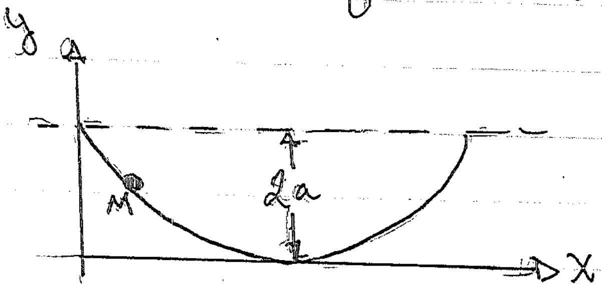
(2). A bead slides without friction on a frictionless wire in the shape of a cycloid described by equations $x = a(\theta - \sin\theta)$ and $y = a(1 + \cos\theta)$ where $0 \le \theta \le 2\pi$. Find a.) the Lagrangian function and b.) the equation of motion. (leave as a differential equation)



(3). Consider a mass m sliding on a smooth inclined plane of mass M, which, itself is free to slide on a smooth horizontal surface as shown in the figure. (In this problem, there are 2 degrees of freedom, so we need two coordinates to specify the configuration. Consider the coordinates x and x' the horizontal displacement of the plane from some reference point and the displacement of the particle from some reference point on the plane, respectively.