cycloid problem 1 1/2

# K Λ H Δ H

a.) cycloid - problem 2

② The curve may have been formed by a cycloidal pendulum of varying amplitude.

(T) or $E_K = \frac{1}{2} m v^2 = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2)$

$$= \frac{1}{2} m a^2 \{[(1-\cos\theta)\dot{\theta}]^2 + [-\sin\theta\,\dot{\theta}]^2\}$$

$$= ma^2(1-\cos\theta)\dot{\theta}^2$$

(V) or $U = mgy = mga(1+\cos\theta)$

$$\mathcal{L} = E_K - U = ma^2(1-\cos\theta)\dot{\theta}^2 - mga(1+\cos\theta)$$

b.) $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$, so

$$\frac{d}{dt}\left[2ma^2(1-\cos\theta)\dot{\theta}\right] - \left[ma^2\sin\theta\,\dot{\theta}^2 + mga\sin\theta\right] = 0$$

$$\frac{d}{dt}\left[(1-\cos\theta)\dot{\theta}\right] - \frac{1}{2}\sin\theta\,\dot{\theta}^2 - \frac{g}{2a}\sin\theta = 0$$

$$(1-\cos\theta)\ddot{\theta} + \frac{1}{2}\sin\theta\,\dot{\theta}^2\ \frac{g}{2a}\sin\theta = 0$$

We can take this a step further by showing that this equation of motion can be written as

$$\frac{d^2u}{dt^2} + \frac{g}{4a} = 0, \quad \text{where } u = \cos(\theta/2)$$

And use this to find the period of oscillation.

Here's how...

If $u = \cos\left(\frac{\theta}{2}\right)$, then

$$\frac{du}{dt} = -\frac{1}{2}\sin\left(\frac{\theta}{2}\right)\dot{\theta}, \quad \frac{d^2u}{dt^2} = -\frac{1}{2}\sin\left(\frac{\theta}{2}\right)\ddot{\theta} - \frac{1}{4}\cos\left(\frac{\theta}{2}\right)\dot{\theta}^2$$

Thus $\frac{d^2u}{dt^2} + \frac{g}{4a}u = 0$ is the same as

$$-\frac{1}{2}\sin\left(\frac{\theta}{2}\right)\ddot{\theta} - \frac{1}{4}\cos\left(\frac{\theta}{2}\right)\dot{\theta}^2 + \frac{g}{4a}\cos\left(\frac{\theta}{2}\right) = 0$$

which can be written as

$$\ddot{\theta} + \frac{1}{2}\cot\left(\frac{\theta}{2}\right)\dot{\theta}^2 - \frac{g}{2a}\cot\left(\frac{\theta}{2}\right) = 0$$

Since

$$\cot\left(\frac{\theta}{2}\right) = \frac{\cos\left(\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} = \frac{2\sin\left(\frac{\theta}{2}\right)\cos\left(\theta/2\right)}{2\sin^2\left(\theta/2\right)} = \frac{\sin\theta}{1-\cos\theta}$$

it follows that $\ddot{\theta} + \frac{1}{2}\cot\left(\frac{\theta}{2}\right)\dot{\theta}^2 - \frac{g}{2a}\cot\left(\frac{\theta}{2}\right) = 0$ is the same as $(1-\cos\theta)\ddot{\theta} + \frac{1}{2}\sin\theta\,\dot{\theta}^2 - \frac{g}{2a}\sin\theta = 0$

The solution of the equation is

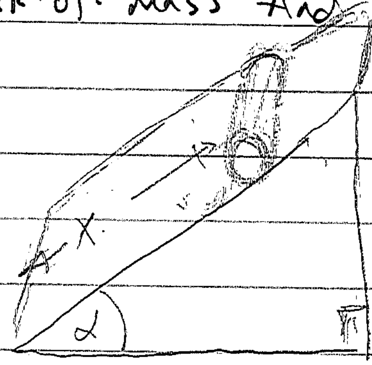
$$u = \cos\left(\frac{\theta}{2}\right) = c_1\cos\left(\sqrt{\frac{4a}{g}}\right)t + c_2\sin\left(\sqrt{\frac{4a}{g}}\right)t$$

from which we see that $\cos\left(\frac{\theta}{2}\right)$ returns to its original value after a time $2\pi\sqrt{\frac{4a}{g}}$ which is the required period. Note that this period is that of a simple pendulum with length $l = 4a$.

Classical Mechanics . . . 4-26-2022

# "WARM UP" Problem

Write down the Lagrangian for a cylinder (mass 'M' radius 'R') rolling without slipping down an inclined plane at an angle "$\alpha$" from the horizontal. Use as your generalized coordinate the cylinder's distance 'x' measured down the plane from its starting point. Write down the Lagrangian equation and solve it for the cylinder's acceleration $\ddot{x}$. Remember that $E_K = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$, where v is the center-of-mass and $\omega$ is the angular velocity.



## THE LAGRANGIAN - ADDENDUM

The Lagrangian is $L = E_k - U$ (or $T - V$, depending on the text/author). Lagrange's equations are

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right)$$

These equations are equivalent to Newton's 2nd Law.

$$\frac{\partial \mathcal{L}}{\partial x} = -\frac{\partial U}{\partial x} = F_x = m\ddot{x} \; ; \; \frac{\partial \mathcal{L}}{\partial \dot{x}} = P_x = m\dot{x}$$

Unlike Newton's 2nd Law, Lagrange's equations have the same form in any coordinates system with generalized coordinates ($q_1, q_2, q_3$). Moreover, we won't have to write out the forces of constraint (e.g., normal force on a bead sliding along a rotating wire.)

$\frac{\partial \mathcal{L}}{\partial q_i}$ = $i^{th}$ component of generalized force

$\frac{\partial \mathcal{L}}{\partial \dot{q}_i}$ = $i^{th}$ component of generalized momentum

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)$$

The path of a particle in 3-dimensional space determined using Newton's Laws is the same as the path determined by the Lagrangian. In 3-dimensional space $L = L(x,y,z,\dot{x},\dot{y},\dot{z})$; note that the equations

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right), \quad \frac{\partial \mathcal{L}}{\partial y} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) \; ; \; \frac{\partial \mathcal{L}}{\partial z} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{z}}\right)$$

have the same form as the Euler-Lagrange equations

$$\frac{\partial F}{\partial x} = \frac{d}{du}\left(\frac{\partial F}{\partial x'}\right), \quad \frac{\partial F}{\partial y} = \frac{d}{du}\left(\frac{\partial F}{\partial y'}\right), \quad \frac{\partial F}{\partial z} = \frac{d}{du}\left(\frac{\partial F}{\partial z'}\right)$$

and imply that the integral $S = \int L\,dt$ is stationary for the path followed by a particle (**Hamilton's Principle**). Stating Hamilton's Principle somewhat more formally "The actual path a particle follows between 2 points 1 and 2 in a given time interval $t_1$ to $t_2$ is such that the action integral $S = \int L\,dt$ is stationary when taken along the path." This is valid for a large class of mechanical systems and for almost any choice of coordinates. Observe that a path can be determined not only by Lagrange's equations (7.7), but also by Hamilton's Principle. (Hamilton's Principle has played an important role in the formulation of quantum mechanics).

We expressed the Lagrangian in cylindrical and spherical coordinates, and based its formulation by substituting appropriately for $q_i$ and $q_i$. Here, $L = \frac{1}{2}mr^2 - U(r) = L(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3)$, and the action integral $S = \int L(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3)dt$.

Remark: The value of the action integral is unaltered by the change of variables.

Lastly, once you've written out $L = E_k - U$, then take some derivatives!

When jumping from on high in a tree,
Just write down del L by del z.
Take del L by z dot,
Then t-dot what you've got,
And equate the results (but quickly!)

# Lagrangian 2 dof

1-26-2022

Problem (in-class) Consider a pendulum made of a spring with mass '$m$' on the end. The spring is arranged to lie in a straight line (which can be arranged, say, by wrapping the spring around a light, massless rod). The equilibrium length of the spring is '$l$'. Let the spring have length $l + x(t)$ and let the vertical angle be $\theta(t)$. Assume that the motion takes place in a vertical plane. Find the equations of motion for $x$ and $\theta$.