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Hamiltonian Mechanics

The Lagrangian method gives us great flexibility in solving problems by using generalized coordinates q_1, q_2, ... q_k, and their derivatives and eliminating forces of constraint. The level of complexity changes considerably, however, for non-conservative systems. Along with Newton's formulation and the Lagrangian formulation, there is yet another formulation we will consider, the Hamiltonian, H. Hamiltonian formalism has physical significance and is better suited for other branches of physics such as quantum mechanics, plasma physics and astrophysics.

The Hamiltonian arises naturally from the Lagrangian. I have included an elegant derivation by Lev Landau and Evgeny Lifshitz.

Recall the Lagrangian is a function of the generalized coordinates $q_1, q_2, \dots q_n$, and their time derivatives (generalized velocities) $\dot{q}_1, \dot{q}_2, \dots \dot{q}_n$, and is written as

$$\mathcal{L} = \mathcal{L}(q_1, \dots q_n, \dot{q}_1, \dots, \dot{q}_n, t) = E_k - U$$

The n coordinates specify a position or configuration of a physical system, and can be thought of as defining a point in an n-dimensional configuration space. When coupled with their derivatives, the generalized coordinates and velocities define a point in **state space**, and specify a set of initial conditions at a time t_0 that determine a unique solution of the n second-order differential equations of motion, Lagrange's equations

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) \quad \{ i = 1, \dots, n \}$$

For each set of initial conditions, these equations of motion determine a unique path or orbit through state pace.

Remark: the terms **state space** and **phase space** have been used interchangeably. Taylor reserves the term state space for the space of generalized coordinates and velocities and phase space for that of positions and generalized momenta.

Recall that the generalized momentum is given by $p_i = \dfrac{\partial \mathcal{L}}{\partial \dot{q}_i}$

If the coordinates $q_1, q_2, \dots q_n$ are Cartesian coordinates, the generalized momenta p_i are the corresponding components of the usual momenta. In general, p_i, is not actually a momentum, but does play an analogous role. The generalized momentum p_i is also called the canonical momentum or the momentum conjugate to q_j.

In the Hamiltonian approach, the Lagrangian is used in the following fashion

$$\mathcal{H} = \sum_{i=1}^{n} p_i \dot{q}_i - \mathcal{L}$$

The equations of motion will involve derivatives of $\mathcal{H}$ rather than $\mathcal{L}$. Note also that with the Hamiltonian, we write the n generalized position coordinates q_n and their generalized momenta p_k, that is

$$(q_1, \dots q_n, p_1, \dots p_n)$$

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... a 2n coordinate phase space

Hamilton's equations for one-dimensional system are written as

$$\dot{q} = \frac{\partial H}{\partial p} \quad \text{and} \quad \dot{p} = -\frac{\partial H}{\partial q}$$

MORE GENERALLY

$$\dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha} \quad \text{and} \quad \dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha}$$

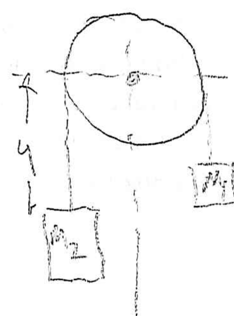
Note that in the Lagrangian formalism, the equation of motion of a one-dimensional system is a second-order differential equation for q, whereas the in the Hamiltonian approach, there are 2 first order differential equations, one for q and for p.

Example: Set up the Hamiltonian formalism for the Atwood machine. Use the height of x of m_1 measured downward as the one generalised coordinate.

recall $\mathcal{L} = E_K - U$

$E_K = \frac{1}{2}(m_1 + m_2)\dot{x}^2$, $U = -(m_1 - m_2)gx - m_2 g l$

$p = \frac{\partial E_K}{\partial \dot{x}} = (m_1 + m_2)\dot{x}$



To solve this to give $\dot{x}$ in terms of p as

$\dot{x} = \frac{p}{(m_1 + m_2)}$ we substitute into H to give

H as a function of x and p:

$$H = E_K + U = \frac{p^2}{2(m_1 + m_2)} - (m_1 - m_2)gx - m_2 g l$$

$x + y = l$

$y = l - x$

mass 1: $U_1 = -m_1 g x$

$U_2 = -m_2 g(l - x)$

$U_1 + U_2 = -m_1 g x - m_2 g l + m_2 g x$

The 2 Hamiltonian equations of motion

$\left(\dot{q} = \frac{\partial H}{\partial p} \text{ and } \dot{p} = -\frac{\partial H}{\partial q}\right)$ are written

as $\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m_1 + m_2}$ and $\dot{p} = -\frac{\partial H}{\partial x} = (m_1 - m_2)g$

so (with a little algebra)

$$\ddot{x} = \left[\frac{m_1 - m_2}{(m_1 + m_2)}\right] g$$

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The formulation of the laws of mechanics in terms of the Lagrangian, and of Lagrange's equations derived from it, presupposes that the mechanical state of a system is described by specifying its generalized coordinates and velocities. This is not the only possible mode of description, however. There are a number of advantages in the study of classical mechanics and quantum mechanics using generalized coordinates and **momenta** of a system. We will derive the form of the equations corresponding to this formulation of mechanics.

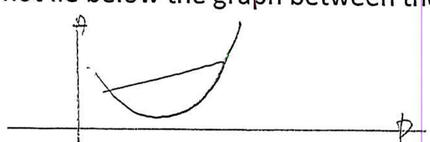
The passage of one set of independent variables to another can be made using Legendre's transformation. Named after Adrien-Marie Legendre, this transformation is an involutive transformation on real-valued convex* functions of one real variable. An involution, involutory function, or self-inverse function is a function that is its own inverse. In physics applications, it is used to convert functions of one quantity (e.g., velocity, pressure, temperature) into functions of a conjugate quantity (momentum, volume, and entropy, respectively). In this way, it is used in classical mechanics to derive the Hamiltonian formalism out of the Lagrangian formalism (or vice versa), in thermodynamics to derive thermodynamics potentials, and in the solution of differential equations of multiple variables.

Any involution is a bijection**.

Examples of an involution include the functions $f_1(x) = -x$, or $f_2(x) = -\frac{1}{x}$, and their composition

$(f_1 \circ f_2)(x) = (f_2 \circ f_1) = f_3(x) = -\frac{1}{x}$. Another example - the graph of an involution (over the real numbers) is line-symmetric over the line y = x. This can be seen by 'swapping' x with y, specifically, if the function is an involution, then it will serve as its own reflection.

*A convex function is a real-valued function if the line segment between any 2 points on the graph of the function does not lie below the graph between the points.



**A function is said to be bijective if f: A → B satisfies both the injective (one-to-one function) and surjective function (onto function) properties. This means that every element 'b' in the codomain B, there is exactly one element 'a' in the domain A such that f(a) = b. If the function satisfies this condition, then it is known as one-to-one correspondence. It is noted that the element 'b' is the image of the element 'a' and element 'a' is the pre-image of the element 'b'.

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In our case, the transformation is developed as follows. The total differential of the Lagrangian as a function of coordinates and velocities is

$$d\mathcal{L} = \sum_i \frac{\partial \mathcal{L}}{\partial q_i} dq_i + \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} d\dot{q}_i$$

This expression may be written as

$$d\mathcal{L} = \sum \dot{p}_i dq_i + \sum p_i d\dot{q}_i$$

This result can be put in another way. Let the Lagrangian be of the form $L = L_o + L'$, where $\mathcal{L}'$ is a small correction to the function L_0. Then the corresponding addition $\mathcal{H}'$ in the Hamiltonian $H = H_o + H'$ is related to L' by $(\mathcal{H}')_{p,q} = -(\mathcal{L}')_{\dot{q},q}$

It may be noticed that, in transforming $d\mathcal{L} = \sum \dot{p}_i dq_i + \sum p_i d\dot{q}_i$ into $dH = -\sum \dot{p}_i dq_i + \sum \dot{q}_i dp_i$

we did not include a term dt to take account of a explicit time-dependence of the Lagrangian, since the time would only be a parameter, which would not be involved in the transformation. Analogously to formula $\left(\frac{\partial \mathcal{H}}{\partial \lambda}\right)_{p,q} = -\left(\frac{\partial \mathcal{L}}{\partial \lambda}\right)_{\dot{q},q}$ the partial time derivatives of L and H are related by

$$\left(\frac{\partial H}{\partial t}\right)_{p,q} = -\left(\frac{\partial \mathcal{L}}{\partial t}\right)_{\dot{q},q}$$

Example 1. Find the Hamiltonian for a single particle in Cartesian, cylindrical and spherical coordinates.

since the derivatives $\frac{\partial \mathcal{L}}{\partial \dot{q}_i}$ are, by definition, the generalized momenta, and $\frac{\partial \mathcal{L}}{\partial q_i} = \dot{p}_i$ by Lagrange's equations. Writing the second term in the equation $d\mathcal{L} = \sum \dot{p}_i dq_i + \sum p_i d\dot{q}_i$ as $\sum p_i d\dot{q}_i = d\left(\sum p_i \dot{q}_i\right) - \sum \dot{q}_i dp_i$

taking the differential $d\left(\sum p_i \dot{q}_i\right)$ to the left-hand side, and reversing signs, we obtain from $d\mathcal{L} = \sum \dot{p}_i dq_i + \sum p_i d\dot{q}_i$

the expression $d\left(\sum p_i \dot{q}_i - \mathcal{L}\right) = -\sum \dot{p}_i dq_i + \sum \dot{q}_i dp_i$

The argument of the differential is the energy of the system expressed in terms of co-coordinates and momenta, it is called the Hamilton's function, or Hamiltonian of the system:

$$\mathcal{H}(p, q, t) = \sum_i p_i \dot{q}_i - \mathcal{L}$$

From the equation in differentials

$$d\mathcal{H} = -\sum \dot{p}_i dq_i + \sum \dot{q}_i dp_i$$

in which the independent variables are the coordinates and momenta, we have the equations

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}$$

These are the required equations of motion in the variables p and q, and are called **Hamilton's Equations.** They form a set of 2n first-order differential equations for the 2n unknown functions $p_i(t)$ and $q_i(t)$, replacing the n second-order equations in the Lagrangian treatment. Because of their simplicity and symmetry of form, they are also called **canonical equations.**

The total time derivative of the Hamiltonian is

$$\frac{d\mathcal{H}}{dt} = \frac{\partial \mathcal{H}}{\partial t} + \sum \frac{\partial \mathcal{H}}{\partial q_i} \dot{q}_i + \sum \frac{\partial \mathcal{H}}{\partial p_i} \dot{p}_i$$

Substitution of q_i and p_i from equations $\dot{q}_i, \dot{p}_i$ shows the last two terms cancel, and so

$$\frac{d\mathcal{H}}{dt} = \frac{\partial \mathcal{H}}{\partial t}$$

In particular if the Hamiltonian does not depend explicitly on time, then $\frac{d\mathcal{H}}{dt} = 0$ and we have the law of conservation of energy.

As well as the dynamical variables q, $\dot{q}$, or q, p, the Lagrangian and the Hamiltonian involve various parameters which relate to the properties of the mechanical system itself, or the mechanical forces on it. Let λ be one such parameter. Regarding it as a variable, we have instead of $d\mathcal{L} = \sum \dot{p}_i dq_i + \sum p_i d\dot{q}_i$

the expression $d\mathcal{L} = \sum \dot{p}_i dq_i + \sum p_i d\dot{q}_i + \left(\frac{\partial \mathcal{L}}{\partial \lambda}\right) d\lambda$

and $d\mathcal{H} = -\sum \dot{p}_i dq_i + \sum \dot{q}_i dp_i$ becomes $d\mathcal{H} = -\sum \dot{p}_i dq_i + \sum \dot{q}_i dp_i - \left(\frac{\partial \mathcal{L}}{\partial \lambda}\right) d\lambda$

Hence, $\left(\frac{\partial \mathcal{H}}{\partial \lambda}\right)_{p,q} = -\left(\frac{\partial \mathcal{L}}{\partial \lambda}\right)_{\dot{q},q}$

which relates the derivatives of the Lagrangian and the Hamiltonian with respect to the parameter λ. The suffixes to the derivatives show the quantities which are to be kept constant in the differentiation.