5-6-2022

Hamiltonian ch. 13 Problems
Solutions

13-(1)

Note $E_K = \frac{1}{2}m\dot{x}^2$

$\mathcal{H} = \sum_{i=1}^{n} p\dot{x} - \mathcal{L} = p\dot{x} - \left(\frac{1}{2}m\dot{x}^2\right) - 0$

$= (m\dot{x})\dot{x} - \frac{1}{2}m\dot{x}^2$

$= m\dot{x}^2 - \frac{1}{2}m\dot{x}^2$

$\mathcal{H} = \frac{1}{2}m\dot{x}^2 = \frac{p^2}{2m}$

$E_K = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2, \quad U = 0$

$\mathcal{L} = E_K - U = \frac{1}{2}m\dot{x}^2$

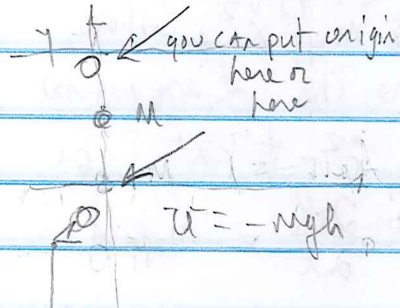
$\mathcal{H} = \sum_{i=1}^{n} p_i\dot{x}_i - \mathcal{L} \longrightarrow \mathcal{H} = p\dot{x} - \mathcal{L} = \frac{p^2}{2m}$

Hamilton's Equations are $\dot{x} = \frac{\partial \mathcal{H}}{\partial p} = \frac{p}{m}, \quad \dot{p} = \frac{-\partial \mathcal{H}}{\partial x} = 0$

Solutions : since $\dot{p} = 0 \Rightarrow p = \text{constant}$

$x = x_0 + v_0 t$ where $v_0 = p_0/m$

13-(2)



$U = -mgh$

$\mathcal{L} = E_K - U$

$\mathcal{L} = \frac{1}{2}m\dot{y}^2 + mgh$

$p = \frac{\partial \mathcal{L}}{\partial \dot{y}} = m\dot{y}, \quad \mathcal{H} = p\dot{y} - \mathcal{L}$

$\mathcal{H} = \frac{p^2}{2m} - mgy$. Hamilton's equations are

$\dot{y} = \frac{\partial \mathcal{H}}{\partial p} = \frac{p}{m}$ and $\dot{p} = \frac{-\partial \mathcal{H}}{\partial y} = mg$

$\dot{y} = \frac{p}{m} \Rightarrow \ddot{y} = \frac{\dot{p}}{m} = g$

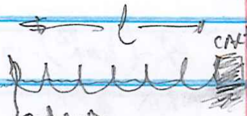
Note, if you write $\mathcal{L} = E_K - U$ with origin here, then $\mathcal{L} = \frac{1}{2}m\dot{y}^2 - mgh$

and $\mathcal{H} = p\dot{y} - \left(\frac{1}{2}m\dot{y}^2 - mgh\right) = m\dot{y}^2 - \frac{1}{2}m\dot{y}^2 + mgh = \frac{1}{2}m\dot{y}^2 + mgh$

$\dot{y} = \frac{\partial \mathcal{H}}{\partial p} = \frac{p}{m}, \quad \dot{p} = -\frac{\partial \mathcal{H}}{\partial y} = -mg$

so, $\dot{y} = \frac{p}{m} \Rightarrow \ddot{y} = -g$ (depending on your frame of reference, not a big deal!!)

13-4) Let the unstretched length of the spring be 'l' and consider a short segment of spring λ from the fixed end of length $d\lambda$. Since the spring is uniform, the mass of the segment is $M \frac{d\lambda}{l}$. Since the spring stretches uniformly, its velocity (when the cart has velocity $\dot{x}$) is $\dot{x}\left(\frac{\lambda}{l}\right)$. Therefore, the kinetic energy of the segment is



$M_{segment} = M\left(\frac{d\lambda}{l}\right)$

$\left(\dot{x}\frac{\lambda}{l}\right)^2 = \dot{x}^2\frac{\lambda^2}{l^2}$

$$E_K = \frac{1}{2} M \dot{x}^2 \lambda^2 \frac{d\lambda}{l^3},$$ and the E_K of the entire spring is

$$E_{K\,spring} = \frac{1}{2}\frac{M\dot{x}^2}{l^3}\int_0^l \lambda^2\, d\lambda = \frac{1}{6} M \dot{x}^2$$

Thus, the total kinetic energy of the cart plus the spring is $E_K = \frac{1}{2} m_{eff}\dot{x}^2$, where $m_{eff} = m + \frac{M}{3}$. Therefore, the Lagrangian for the system is $\mathcal{L} = \frac{1}{2} m_{eff}\dot{x}^2 - \frac{1}{2}kx^2$, the generalized momentum is $p = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m_{eff}\dot{x}$

and the Hamiltonian is $\mathcal{H} = p^2/(2m_{eff}) + \frac{1}{2}kx^2$.

The two Hamiltonian equations are

$$\dot{x} = \frac{\partial \mathcal{H}}{\partial p} = \frac{p}{m_{eff}} \quad \text{and} \quad \dot{p} = -\frac{\partial \mathcal{H}}{\partial x} = -kx.$$

Combining, we find the $\ddot{x} = \frac{\dot{p}}{m_{eff}} = \frac{-kx}{m_{eff}}$;

Solving $\ddot{x} + \frac{kx}{m_{eff}} = 0$, we find that the cart oscillates with frequency $\omega = \sqrt{\frac{k}{m_{eff}}}$

13-10)

given $\vec{F} = -kx\,\hat{x} + \bar{K}\,\hat{y}$; $d\vec{r} = \hat{i}\,dx + \hat{j}\,dy$; let $\hat{x}=\hat{i}$, $\hat{y}=\hat{j}$

proceeding on this basis, $\vec{F}\cdot d\vec{r} = -kx\,dx + \bar{K}dy$

by definition, $U = -\int \vec{F}\cdot d\vec{r}$ $\Rightarrow dU_x = -(-kx)\,dx = kx\,dx$ $\quad dU_y = -\bar{K}\,dy$

$\left(\text{or } -U = \int \vec{F}\cdot d\vec{r} \right)$

so, $U_x = \frac{kx^2}{2}$ $\quad U_y = -\bar{K}y$

$$E_k = \frac{1}{2}m(v_x^2 + v_y^2) = \frac{1}{2}mv_x^2 + \frac{1}{2}mv_y^2$$

recall $v_x = \dot{x}$ & $v_y = \dot{y}$

so, $E_K = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2$

This is alot of algebra! ☹ (sorry!) but I've tried to include all of my steps

Recall $P_x = m\dot{x}$ $\quad P_y = m\dot{y}$

Now, $E_{K_x} = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}\left(\frac{P_x^2}{m}\right)$ $\quad E_{K_y} = \frac{1}{2}m\dot{y}^2$, $E_{K_y} = \frac{1}{2}\left(\frac{P_y^2}{m}\right)$

$$E_{K_x} = \frac{P_x^2}{2m} \qquad E_{K_y} = \frac{P_y^2}{2m}$$

Therefore

$$\mathcal{H} = \sum_{i=1}^{n=2} p_i \dot{q}_i - \mathcal{L}$$

$$= P_x\dot{x} + P_y\dot{y} - \left[\underbrace{\left(\frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2\right)}_{E_K} - \underbrace{\left(\frac{kx^2}{2} - \bar{K}y\right)}_{U} \right]$$

$$= P_x\dot{x} + P_y\dot{y} - \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\dot{y}^2 + \frac{kx^2}{2} - \bar{K}y$$

$$= m\dot{x}\dot{x} + m\dot{y}\dot{y} - \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\dot{y}^2 + \frac{kx^2}{2} - \bar{K}y$$

$$= m\dot{x}^2 + m\dot{y}^2 - \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\dot{y}^2 + \frac{kx^2}{2} - \bar{K}y$$

$$= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 + \frac{kx^2}{2} - \bar{K}y$$

Substituting, we obtain

$$\mathcal{H} = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2}kx^2 - K y$$

The two Hamiltonians are

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{2 p_x}{2m} = \frac{p_x}{m} \qquad \dot{p}_x = \frac{-\partial H}{\partial x} = -kx$$

$$\Rightarrow \ddot{x} = \frac{\dot{p}_x}{m} = -\frac{kx}{m}$$

$$\text{or,} \quad \ddot{x} + \left(\frac{k}{m}\right)x = 0$$

Solving, $x = A\cos(\omega t - \varphi)$ with angular frequency

$$\omega = \sqrt{\frac{k}{m}}$$

all good! This agrees with Taylor ☺

For y : $\dot{y} = \frac{\partial H}{\partial p_y} = \frac{p_y}{m}$ And $\dot{p}_y = \frac{-\partial H}{\partial y} = +K$

which combine to give $\ddot{y} = \frac{K}{m}$