Potentials and Fields

So far you have learned that there is a remarkable similarity between electric forces and gravitational forces. You have also learned that we can deal with both of these "action-at-a-distance" forces by introducing the concept of the field. The charge (or mass) at one location creates an electric (or gravitational field) at all points in space. Another charge (or mass) reacts to this field, experiencing an electric (gravitational) force.

Today we will finish out the analogy by relating these forces to potential energies. Along the way, we will introduce a useful new concept, the **potential** (which is closely related to, but not the same as, potential energy).

Potential Energy

Recall from last semester that the gravitational force is **conservative**. This means that as we move an object around in space, the total work done by the gravitational force on the object is *independent of the path*.

For example, suppose a student needs to get from SSC130 to the computer lab on the second floor of the Stevens Science Center. She could simply walk up the front (east staircase). Alternatively, she could go down to the greenhouse on the ground floor to check her botany project, walk up the south staircase to the third floor to meet with Dr. Stultz, then walk back down the east stairway to the computer lab. In both paths, work done by the gravitational force is the same: $W_g = -mg\Delta h$, where Δh is the vertical displacement from the initial to the final position. The negative sign comes from the fact that the (downward) gravitational force is directed opposite to the (upward) displacement.

The change in gravitational potential energy, $\Delta U_g = mg\Delta h$, is defined to be the negative of the work done by the gravitational force. In both cases above, the change in gravitational potential energy is the same. (If the student took the longer path, she object that she definitely expended more energy in walking around the building and up and down floors, but that's due to work done by other (usually dissipative) forces. The work done by gravity and the change in potential energy are the same for the two paths.)

Of course the formula $\Delta U_g = mg\Delta h$ is only true when the gravitational field is uniform, which is largely true near Earth's surface. For distances over which the gravitational field strength aid direction changes, we would need to integrate the gravitational force to get the change in potential energy between two points.

For two spherical or point-like masses separated by a distance r, such as the sun and a planet in orbit around it, the gravitational potential energy is

$$U_g = -\frac{Gm_1m_2}{r} \ .$$

Notice that this gives a specific separation for zero potential energy: $U_g = 0$ as $r \longrightarrow \infty$. In our previous studies of gravitational potential energy, we were free to define U_g to be zero at any convenient point. This is still the case; only the difference in potential energy is a relevant physical quantity. However, for forces between point masses it turns out most convenient to define the potential energy to vanish at infinite separation.

The negative sign in the formula above guarantees that for every separation shorter than infinity, the potential energy decreases as the objects get closer. This is related to the fact that gravity is an attractive force: *the force of gravity points in the direction of decreasing gravitational potential energy*.

You will not be surprised to learn that we can similarly define potential energy between spherical or point-like electric charges. As before, the potential energy will depend on the product of the charges q_1 and q_2:

$$U_e = \frac{k_eq_1q_2}{r} \ .$$

As with gravity, the potential energy is defined to approach zero as the separation becomes infinite. However, notice the difference in signs. The potential energy is negative when the product q_1q_2 is negative (opposite charges), and decreases as the charges get closer together. The potential energy is positive when the product q_1q_2 is positive (like charges), and decreases as the charges get farther apart. Again, this is consistent with the idea that *forces point in the direction of decreasing potential energy*. For like charges the force is repulsive, while for opposite charges the force is attractive.

Gravitational Potential

Go back to the example of a student walking upstairs to the computer lab (in a uniform gravitational field). Compare her change in gravitational potential energy to that of a much larger student who started and ended in the same places. The change in potential energy will be different for each student, but only because they have different masses. What is the same for both students is the quantity

$$\Delta V_g = \frac{\Delta U_g}{m} = g\Delta h \ ,$$

where V_g is known as the **gravitational potential**. This quantity only depends on the change of an object's position in a gravitational field.

Unlike gravitational potential energy, which is associated with two masses, gravitational potential is associated with a single mass. This raises an obvious analogy: *potential is to potential energy as field is to force*. This is easy to see with the expressions for spherical or point-like masses:

$$V_g = -\frac{Gm_1}{r}\,, \quad U_g = -\frac{Gm_1 m_2}{r}\,, \quad \vec{\mathbf{g}} = -\frac{Gm_1\hat{\mathbf{r}}}{r^2}\,, \quad \vec{\mathbf{F}}_g = -\frac{Gm_1 m_2\hat{\mathbf{r}}}{r^2}\,.$$

We can also rearrange the analogy: just like force points in the direction of decreasing potential energy, *field always points in the direction of decreasing potential*.

Electric Potential

Again, you will not be surprised to learn that there is an electric potential that is completely analogous to gravitational potential. If we move a test charge q in an electric field, the change in potential is related to the change in potential energy of the test charge:

$$\Delta V_e = \frac{\Delta U_e}{q}\,.$$

The units of electrical potential are joules (energy) per coulomb (charge). We call this new unit the **volt** (V), $1\,\mathrm{V} = 1\,\mathrm{J/C}$. This also tells you something about the meaning of "voltage" of a battery. For example, in a 9 V battery, the positive terminal of the battery has an electrical potential that is 9 volts greater than the potential at the negative terminal of the battery. Most other batteries you encounter, sizes AAA through D, have a 1.5 V potential difference between the positive and negative terminals of the battery.

For a spherical or point-like charge q_1, the electric potential at a distance r from the charge is

$$V_e = \frac{k_e q_1}{r}\,.$$

Just like with gravity, *electric field always points in the direction of decreasing electrical potential*. It points away from a positive charge ($V_e \longrightarrow 0$ as $r \longrightarrow \infty$) and toward a negative charge ($V_e \longrightarrow -\infty$ as $r \longrightarrow 0$).

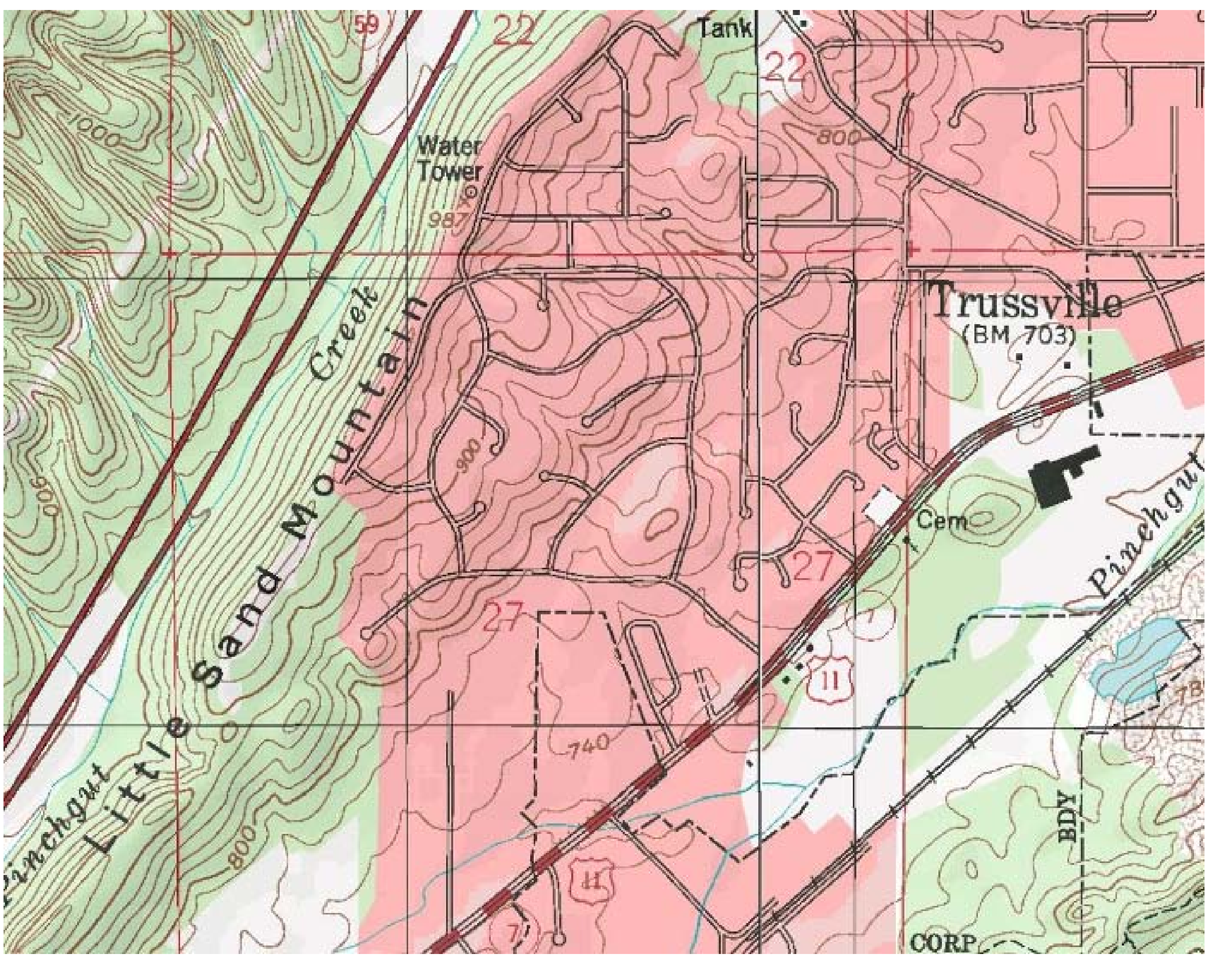
Contours, Level Surfaces, and Equipotential Lines

If you have any experience with topographical maps (see diagram below), you have already encountered equipotential lines. The picture below shows a topographical map of an area near my house in Trussville. At the center of the map between US Highway 11 (red and white

striped line in the lower-right corner) and I-59 (double red line in the upper-left corner) is a neighborhood built on the southeastern side of a ridge known as “Little Sand Mountain”. The red curves in the map are lines of *constant elevation above sea level*, and are spaced in 20 foot intervals. The highest elevation in the neighborhood, at 987 ft above sea level, is near the water tower. North-west of I-59 is the much higher Red Mountain.

The highest points on Little Sand Mountain run southwest to northeast in the map. On the neighborhood side, the elevation decreases slowly to US-11. On the other side, the elevation decreases rapidly to I-59. As you may have learned in scouting, the more closely spaced the elevation lines, the steeper the hill. The hill is very steep on the interstate side and less steep on the neighborhood side.

So what does this have to do with gravitational or electrical potential? Earth’s gravitational field is uniform near Earth’s surface, so we can write $V_g = gh$, where h is elevation above sea level. (I have chosen to set $V_g = 0$ at sea level.) The the contours on the map are lines of constant gravitational potential. We call these **equipotential lines**.



Because we are plotting the equipotential lines on a surface (the ground, roads, etc.) The relevant components of the gravitational field and gravitational force are the ones that point along the surface. This means that if we place a ball on top of the mountain and give it a push one way or the other, it will experience a force pulling it downhill—*toward decreasing potential energy*. The gravitational field, therefore, points downhill *toward decreasing potential*. The more closely spaced the equipotential lines, the greater the field strength. If we were to put identical balls on top of the mountain and let them fall in either direction, the one on the steeper side will experience a greater component of gravitational force pulling it downhill than will the one on the shallower side.

There is an important lesson to be learned from this analysis. The directions "uphill" and "downhill" point perpendicular to the contour lines. What you need to take away is this: *fields always point perpendicular to equipotential lines, and in the direction of decreasing potential*.

We also talk about equipotential lines or equipotential surfaces for electric potential. For a point charge q, since the electric potential is $V_e = k_e q/r$, the surfaces of constant potential are spheres centered on the charge. The electric field, therefore, will point perpendicular to these surfaces, either radially outward (from a positive charge) or radially inward (toward a negative charge) as determined by the direction of decreasing potential. The equipotential surfaces are more closely spaced near the charge (where $1/r$ changes rapidly) and less closely spaced far from the charge (where $1/r$ changes slowly). This means that the electric field magnitude is greater near the charge, and weaker far from the charge, as we have already seen.

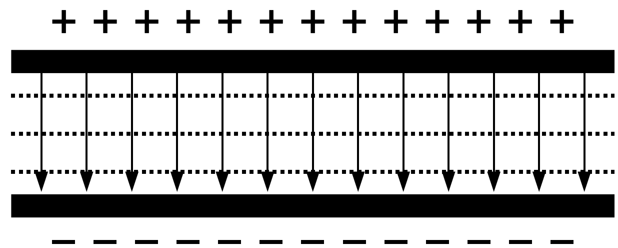
A Note About Directions

In summary, we have two important facts:

- Whenever there is a conservative force, there will be associated with that force a potential energy. The force always points in the direction of decreasing potential energy.
- Whenever there is a conservative field (a field associated with a conservative force), there will be associated with that field a potential. The field always points in the direction of decreasing potential.

These two facts are not confusing for gravity, where there is only positive mass. However, it can be confusing for electricity, where there can be positive and negative charges.

The diagram below shows a good example, in which we have two oppositely charged, closely spaced plates.



The electric field $\vec{\mathbf{E}}$ between such plates is uniform (the downward arrows) because the equipotentials (dotted lines) are uniformly spaced. This makes it very analogous to the gravitational field near Earth's surface. The electric potential is higher at the positive plate and lower at the negative plate. If the plate separation is d, then the potential difference between the plates is $\Delta V = Ed$, where E is the (constant) magnitude of the electric field between the plates. The electric field points downward, from high potential (positive plate) to low potential (negative plate). Let's say the potential difference between the two plates is 1 V. Because we can, let's choose to make the potential of the negative plate be $V_{\text{bottom}} = 0\,\text{V}$. Then the potential on the positive plate will be $V_{\text{top}} = +1\,\text{V}$.

If we place a positive charge $q = +1\,\text{C}$ between the plates, it will experience a downward electric force. We can explain this two different ways. First, the positive charge will be repelled by the positive plate and attracted to the negative plate. Second, the potential energy of the charge, $U_e = +qV_e$, will be 0 J on the bottom (negative) plate, and +1 J at the top (positive) plate. The electric force on the charge will point in the direction of decreasing potential energy.

If we place a negative charge $q = -1\,\text{C}$ between the plates, it will experience an upward electric force. Again, we can say that the negative charge will be repelled by the negative plate and attracted to the positive plate. We can also say that the potential energy of the charge will still be 0 J on the bottom (negative) plate, but will now be -1 J on the top (positive) plate. The potential energy is lower at the positive plate and higher at the negative plate, ensuring that the force on the charge points points upward, opposite the electric field.