# Probability Theory

## Percentiles

**Def:** The **median** of a continuous random variable is the point $x$ such that exactly half of the probability for $X$ is at or below $x$. I.e., the median is the value $x$ such that $F(x) = 0.5$.

The median for a discrete random variable is not well-defined.

For a sample of $n$ points, the median is the middle value of the sorted list of points if $n$ is odd, and it is the average of the two middle values if $n$ is even. (Thus the median of a sample need not be an actual point in the sample).

**Ex:** Find the median of the sample $1, 5, 1, 6, 6, 2, 3, 8$.

In general:

- The $100p^{th}$ *percentile* for a continuous r.v. is the value $x$ such that $F(x) = p$.
- The $50^{th}$ percentile is the median.
- The $25^{th}$ percentile is the **first quartile**.
- The $75^{th}$ percentile is the **third quartile**.

- The lowest value with positive probability (or density) is the **minimum**.
- The highest value with positive probability (or density) is the **maximum**.
- The maximum minus the minimum is the **range**.
- The third quartile minus the first quartile is the **inner quartile range**.
- The average of the maximum and minimum is the **midrange**.
- If $g$ is a strictly increasing function, then the $100p^{th}$ percentile of $g(X)$ is the same as for $X$.
- If $g$ is a strictly decreasing function, then the $100p^{th}$ percentile of $g(X)$ is the same as the $100(1-p)^{th}$ percentile for $X$.

**Ex:** The loss size for an automobile insurance policy has density

$$f(x) = \begin{cases} \frac{3 \cdot 500^3}{(500+x)^4} & x > 0 \\ \\ 0 & \text{otherwise} \end{cases}.$$

Determine the $75^{th}$ percentile for losses on the policy.

**Ex:** A random variable has the following distribution function:

$$F(x) = \begin{cases} 0 & x < 0 \\ x^a & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases} .$$

The $30^{th}$ percentile of the distribution is 0.35. Calculate the median.

**Ex:** Let $X$ be a continuous random variable with density

$$f(x) = \begin{cases} \frac{1}{3}e^{-x/3} & x > 0 \\ 0 & \text{otherwise} \end{cases} .$$

Determine the $35^{th}$ percentile for the random variable.

**Ex:** Let $X$ be a continuous random variable with density

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & \text{otherwise} \end{cases}.$$

Determine $\lambda$ if the median of the distribution is $\frac{1}{4}$.

**Ex:** Losses on an insurance policy have the following distribution function:

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-x/1000} & 0 \leq x \end{cases}.$$

If the policy has a deductible of 50, determine the median payment amount given that a payment was made.