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Linearizing Mile Run Times

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In 1975, Steve Elliot ran the mile in $T := 247.4$ seconds, setting the record for Michigan high school runners [**1**]. By 1982, Michigan high schools had replaced the mile event with the 1600 meter race. A mile is approximately 1609 meters (actually, one mile $= 5280$ feet $\times\ 12\frac{\text{inches}}{\text{feet}} \times .0254\frac{\text{meters}}{\text{inch}} = 1609.344$ meters, but 1609 will do for our purposes); so when Earl Jones in 1982 ran 1600 meters in 247.2 seconds [**1**], a natural question arose. Since the former time linearizes to a lower time of 246.0 $(= T \times \frac{1600}{1609})$ for 1600 meters, who should have been considered the state record holder for 1600 meters just after the latter race? Our analysis will not yield a definitive answer to this question, but will provide a basis for further discussion.

What follows is a conversation between the eponymous Lynn, who believes that linearizing should be done so that the record goes to Elliot, and her counterpart Nolyn, who thinks that one must run in a race of the actual distance to hold the record, so the record should belong to Jones. The question is clearly equivalent to that of which runner should be considered as having achieved the higher average speed over a distance of 1600 meters. The average speed of Elliot's mile run was $\frac{1609}{247.4}$, which is greater than $\frac{1600}{247.2}$, the average speed of Jones's 1600 meter run.

Lynn: "The issue is whether Elliot should be given credit for running 1600 meters at the average speed of $V := \frac{1609}{247.4}$. The mean value theorem states that he must have attained an instantaneous speed of V at some time in the race. This could be stated as saying that Elliot had an average speed of V over a certain infinitesimal interval of time. Must there also be a subinterval of $[0, 247.4]$ corresponding to a distance

interval of 1600 meters over which Elliot ran at the average speed V? Let Elliot's run be described by the function $t(s)$, with the time t being given in terms of the distance s. Then t is continuous, with $t(0) = 0$ and $t(1609) = 247.4$. The function $t^*(s) = t(s) - \frac{247.4}{1609}s$ may be extended from $[0, 1609]$ to a continuous function on $\mathbb{R}$ which is periodic of period 1609, since $t^*(0) = t^*(1609) = 0$. The integral over each period is the same, so

$$\begin{aligned} 0 &= \int_0^{1609} t^*(s + 1600)\,ds - \int_0^{1609} t^*(s)\,ds \\ &= \int_0^{1609} t(s + 1600) - t(s) - \frac{247.4}{1609}(s + 1600 - s)\,ds. \end{aligned}$$

By the mean value theorem for integrals, the integrand must be zero for some s_0. Thus

$$\frac{1600}{t(s_0 + 1600) - t(s_0)} = \frac{1609}{247.4},$$

and so Elliot ran the interval $[s_0, s_0 + 1600]$ at an average speed V.

Nolyn: "If $s_0 \in (9, 1609)$ the 'interval' $[s_0, s_0 + 1600]$ is not an interval run by Elliot, but is equivalent to $[0, s_0 - 9] \cup [s_0, 1609]$. Figure 1 shows two possible ways of running a mile in $T = 247.4$ seconds.

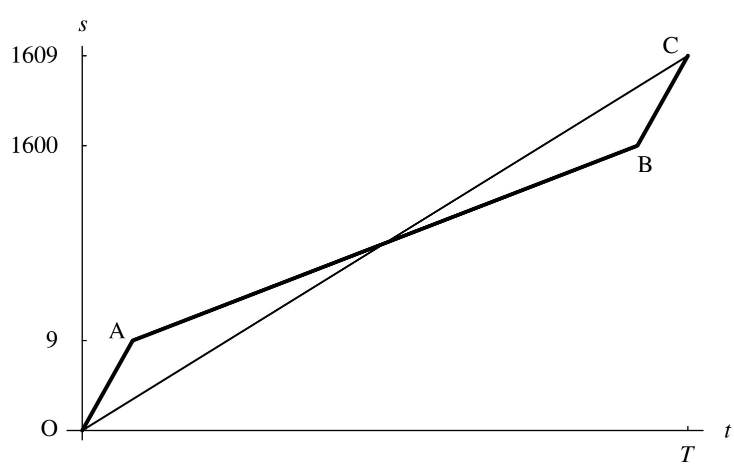

Figure 1.

"The thin line represents running at constant speed V, while the thick line corresponds to running the first and last 9 meters at constant speed V_1 and the middle 1591 meters at constant speed V_2, where $V_1 > V > V_2$. If the mile was run according to the thick line, then the average speed corresponding to the distance interval $[0, 1600]$ is the slope v of the line segment OB and hence is less than the thin line's slope V. But for any $s \in [0, 9]$, the corresponding line segment is parallel to OB and so the average speed over $[s, 1600 + s]$ is also v. Thus the runner has run every subinterval of length 1600 at a lower average speed than his overall average speed."

Lynn: “So my conjecture fails in general. The essence of your counterexample is two short bursts of extremely high speed, one at each end of the race. But this is unrealistic in that it assumes a high speed at the very beginning, whereas runners have to start from rest.”

Nolyn: “Modify the dark path by keeping the point A at the same height of 9 meters, but move it to the right of the thin line. Then keep B at a height of 1600 meters, but move it to the left of the thin line enough to keep BC parallel to OA. See Figure 2.

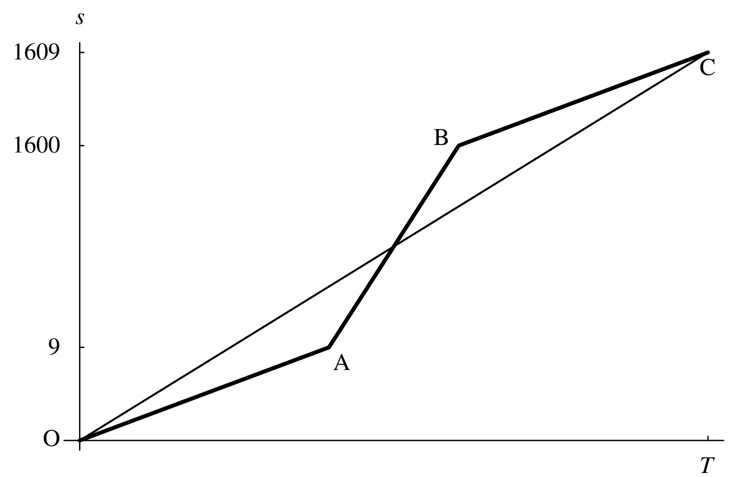

Figure 2.

“As before every average speed corresponding to an interval of length 1600 is equal to the slope of OB and this slope is strictly greater than V.”

Lynn: “But if the runner runs all these 1600s at a speed even greater than V, surely he should be given credit for running one 1600 at the speed V. Must there be an interval of length 1600 with a corresponding average speed of V or more whenever the average speed is low for a short initial interval?”

Nolyn: “I cannot contradict this with an example made of 3 line segments. However, return to the original picture (Figure 1) and note that the reasoning still holds if the congruent line segments OA and BC are replaced by any congruent pair of increasing graphs, the first connecting O to A and the second connecting B to C. Now, as Figure 3 shows, it is easy to have a slow start, and yet all the average speeds corresponding to intervals of length 1600 are still equal to v and hence less than V.”

We leave the discussion at this point, although we are sure that Lynn will not give up so easily.

Remark 1. *Looking at Figures 1 and 3 confirms that we can still have a counterexample after further smoothing the curves passing from O to A and from B to C. So it is easy to construct an infinitely differentiable example. There is something special about* 1600 *and* 1609*; it is important that* 1600 *is more than half of* 1609*. For example, it is easy to show that either the first half mile or the second half mile is covered at a rate as least as large as the overall rate.*

Remark 2. *When comparing times between two similar distances, the effect of the initial start becomes much more significant when the distances are shorter. For ex-*

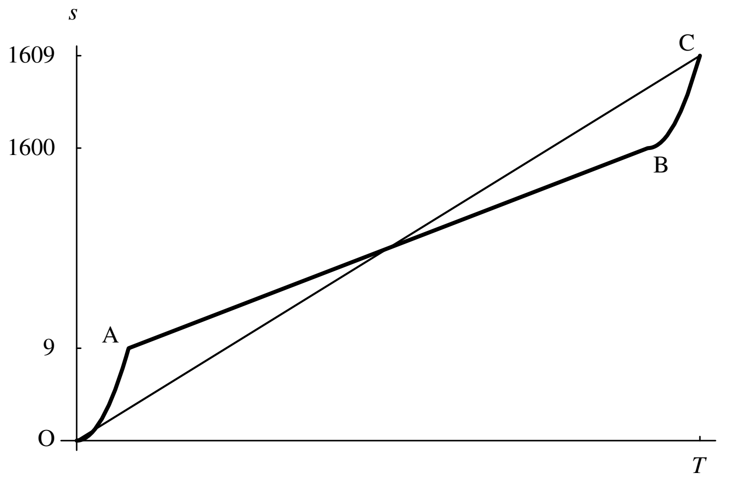

Figure 3.

ample, at the present time Michael Johnson's 200 *meter world record of* 19.32 *seconds is less than twice Tim Montgomery's* 100 *meter record of* 9.78 *seconds, but we do not think that anyone is seriously proposing that the* 100 *meter record be given to Johnson.*

Metric-English conversion

John H. Conway told us about some interesting facts connected with conversion between the metric and English systems. In the early part of the twentieth century, the Anglo-Americans held sway and the world conversion rule was 1 meter = 39.37 inches. Subsequently, Australia, Canada, New Zealand, South Africa, the United Kingdom, and the United States agreed that effective July 1, 1959, the standard would be metric in the sense that the conversion rule would change to 1 inch = 2.54 cm. One day, while being driven from Alabama to Atlanta, Conway noted a succession of kilometer and mile markers. This caused him to make all of the following observations. First, 127 plays a curious role. We have $2.54 = 127 \times .02$ and $39.37 = 127 \times .31$, and $127^2 \times 62 = 999998$. One consequence is that if one million meters be converted to inches using the old rule, and then converted back to meters using the new one, exactly 2 meters are lost. Another is that if the area of a 20 cm by 20 cm square is computed by using the old rule in one dimension and the new rule in the other, then the area is exactly 62 in^2. Turning now to miles and kilometers, which were noted above to be in the ratio of 1 to 1.609344, he noted that 64 miles is almost exactly 103 kilometers ($103/64 = 1.609375$ which is very close; in part because $103/64 = \langle 1, 1, 1, 1, 1, 3, 1, 2\rangle$ is the seventh convergent of $\langle 1, 1, 1, 1, 1, 3, 1, 2, 7, 1, 1, 15\rangle$, the continued fraction of 1.609344) and also that a nice conversion method for distances under 103 kilometers is to write the number as a sum of a small number of Fibonacci numbers, and then bump the summands up or down one term in the Fibonacci sequence depending on whether you want to go from miles to kilometers or vice versa. (The first few Fibonacci numbers are 1, 1, 2, 3, 5, 8, 13, 21, 34, and 55. For example, $47 = 34 + 13$. Bumping up converts 47 miles to $55 + 21 = 76 \approx 75.639$ kilometers while bumping down converts 47 kilometers to $21 + 8 = 29 \approx 29.204$ miles.) This works pretty well because the golden mean 1.618 ... is close to 1.609 and successive Fibonacci numbers have a ratio close

to the golden mean. Combining the 64 $\longleftrightarrow$ 103 rule and the Fibonacci method covers all earth distances quite nicely since the largest possible such distance is only about 20,000 kilometers.

Reference

1. Michigan High School Track & Cross Country, internet URL: http://michtrack.org/ATL/Boys%20Outdoor%20ATL-running.pdf, 2003.

Concerning "A Serendipitous Proof"

Man Keung Siu (mathsiu@hkucc.hku.hk) of the University of Hong Kong writes:

> Readers of David Perkins's interesting article "A Serendipitous Proof" (*CMJ*, November (2003) 359–361) may be interested in learning that the solutions presented there also appeared in the ancient Chinese mathematical classic *Jiu Zhang Suan Shu* (Nine Chapters of the Mathematical Art). This work is believed to have been compiled sometime between 100 BC and 100 AD. The formula in Perkins's second solution appears in Problem 16 of Chapter 9, and in a commentary by Liu Hui (ca. 250 AD), it is explained through a dissection method. The commentary contains several other formulas, including the one in Perkins's first solution and one more of interest: $d = \sqrt{(c-a)(c-b)}$. Relationships between the various formulas and their implications are provided in Man Keung Siu, Proof and Pedagogy in Ancient China: Examples from Liu Hui's Commentary on Jiu Zhang Suan Shu, *Educational Studies in Mathematics*, **24**(1993) 345–357.