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Sudoku: Just for Fun or Is It Mathematics?

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Sudoku, a logic puzzle involving numbers, is the newest sensation sweeping the nation. When you read about Sudoku, almost every author states that there is no math involved in solving the puzzle. This is likely to defray the anxieties of the math anxious population, but is it true?

The most common type of Sudoku puzzle consists of a 9×9 grid with nine 3×3 subgrids or regions. A puzzle is presented by providing a grid partially filled with the digits 1 to 9, and the object is to fill the rest of the cells so that every row, column, and region contain the digits 1 to 9 once and only once (see Figure 1). In a well-formed puzzle there is only one possibility for each empty cell and thus the puzzle has a unique solution. Many people have perfected logic techniques for solving these puzzles as done in online puzzle sites such as `www.brainbashers.com/sudoku.asp`.

2	6		7			3		9
8			2		5	7		
	1						7	
4		2				6		3
	3						2	
7		3	1		4	9		2
		4	5					
		6	9		2		3	8

Figure 1. (created using `www.dailysudoku.co.uk/sudoku/play.shtml`)

Sudoku has a disjointed history, but its roots are in puzzles called magic squares, which date to China in 2800 B.C. Magic squares consist of a square grid filled with distinct positive integers arranged so that the sum of the numbers in any horizontal, vertical, or main diagonal line is always the same number. The Europeans were introduced to magic squares when they started to trade steadily with Asia around the 8th century A.D. By the 13th century they were altered by the Italians who formed a new type of puzzle called Latin squares. An $n \times n$ Latin square consists of the numbers 1 to n arranged in each row and column so that no row or column contains the same number twice. Traditional Sudoku puzzles are a special case of Latin Squares with $n = 9$ and the additional restriction that particular 3×3 regions must also contain each of the digits 1 to 9.

Contrary to its name origin, the concept of Sudoku was not invented in Japan. This particular type of puzzle is believed to have been created by Indiana architect Howard Garns as Number Place puzzles published in the U.S. in the late 1970s. In 1984 a Japanese puzzle company presented them to their puzzle fans and the popular puzzle became known as Sudoku. The modern Sudoku craze began near the end of 2004 when Sudoku puzzles were published in *The London Times*. Now there are Sudoku websites, competitions, chat rooms, books, and more. Sudoku is the fastest growing puzzle in the world and has earned its title as "the Rubik's cube of the 21st century."

Counting Puzzles

Finding a way to count all possible completed 9×9 Sudoku grids is no easy task. Perhaps the easiest way is to break down the problem by factoring it into a more easily calculated one. Two mathematicians Bertram Felgenhauer and Frazer Jarvis described this approach in their paper entitled "Enumerating possible Sudoku grids." They took advantage of the fact that numerous Sudoku grids could be created through simple relabelling. These factors allowed them to create a computer program to determine the rest through a series of loops at a

relatively low run time. The total number of possible Sudoku grids is approximately 6.671×10^{21}. Another paper by Jarvis and Ed Russell shows that there are *only* about 5 billion "essentially different" Sudoku grids when symmetry is taken into account.

Mathematizing Sudoku

Beyond counting the number of Sudoku grids and the logic of solving Sudoku puzzles there are many mathematical questions that arise from considering the puzzle and many of these would be great projects for students conducting undergraduate research. These questions arise in a variety of ways.

One area of exploration is the mathematical formulations of the definition of a Sudoku puzzle and considerations of the resulting structures. For instance, one way we can think of the 9×9 version of the Sudoku puzzle is as a function that assigns each ordered pair (x, y) (where $1 \le x, y \le 9$) a number between 1 and 9 such that if (x, y) and (x', y') are distinct ordered pairs then $S(x, y) \ne S(x', y')$ if the following conditions are satisfied:

1. $x = x'$, or
2. $y = y'$, or
3. $\lceil x/3 \rceil = \lceil x'/3 \rceil$ and $\lceil y/3 \rceil = \lceil y'/3 \rceil$.

This formulation can then be used to create conjectures and proofs about the function S. For instance we can show that S must be a one-to-one and onto function. Alternatively, this could also be used to create a graph theoretic view of the puzzle where two vertices are connected by an edge if they are in the same row, column or 3×3 region, and the puzzle could be studied using graph theory.

Another rich source of mathematical exploration with Sudoku is developing results based on the logic techniques that are used to solve the puzzle. For instance, the rules of Sudoku can be used to make conjectures that address when a certain cell's identity can be guaranteed using knowledge of other cells. An example of one of these results is the following proposition.

Proposition: For a given cell in a 3×3 region, if all other cells outside the region in the same row and column are known, then the missing cell can be determined regardless of the value of the other cells in the region.

Proof: Let (x, y) be the cell whose value is to be determined. We will say that it is in row X, column Y and region R, and then by assumption we know all the values of the cells in row X and column Y that are outside region R. We will use proof by contradiction to show that the value for (x, y) is unique.

Suppose that the value of the missing cell (x, y) cannot be determined from this information alone, i.e., there are at least two distinct digits from $\{1,\ldots,9\}$ that could be placed in the missing cell. Call them a and b. Then row X and column Y are both missing a and b. Since at most one of them, say a, can be placed in the intersection cell (x, y) then b must be placed in row X and in column Y somewhere other than their intersection. But this would force b to appear twice in region R, for a contradiction. So the value for (x, y) is unique. In fact, since the value for (x, y) is unique and can be determined from this information alone, there must be exactly one digit missing from row X and column Y. Q.E.D.

				1				
				2				
				3				
2	4	5				7	8	1
				7				
				8				
				9				

Figure 2. The shaded square must be completed with 6.

As an example, in the partial Sudoku puzzle given in Figure 2, the middle column is missing the digits 4, 5, and 6, while the middle row is missing the digits 3, 6, and 9. Since 6 is the only number needed by both, it must be in the highlighted cell.

Sudoku Variants

Restrictions and extensions of traditional Sudoku definitions are other methods that provide rich sources for problem explorations. Some people, including the original Japanese publisher of Sudoku, consider a necessary part of a Sudoku puzzle to be the added constraint that the placement of the given digits appears in a pattern with 180° rotational symmetry. This means that in the presentation of a Sudoku puzzle if a given digit is in cell (x, y) then there will also be a given digit in cell $(10 - x, 10 - y)$, the position that arises from a 180° rotation of the puzzle. This constraint is for aesthetically pleasing reasons, not for reasons of logic. Consider the Sudoku given in this article (Figure 1). It does not have such symmetry but still leads to a unique logical solution. This symmetry restriction can lead to other research questions, including questions regarding the number of puzzles with this added constraint.

Similar to the way in which Sudoku was created as a special case of Latin squares, a specific class of Sudoku puzzles can be created by extending the row, column, and 3×3 region restrictions to the diagonals. For an example of this, after you solve the puzzle given in Figure 1, take a closer look at the diagonals. Knowing that this restriction was added makes

solving the puzzles easier, but it adds complexity to creating them. How many Sudoku puzzles are there with this restriction?

Extensions to 9×9 Sudoku puzzles can be made by considering general $n^2 \times n^2$ grids with $n \times n$ regions to be filled with the digits 1 to n satisfying Sudoku-type constraints. Alternatively extensions can be made by considering any $nm \times nm$ grid with $n \times m$ regions such as a 6×6 grid with 6 individual 2×3 regions. Puzzle makers sometimes refer to these as super Sudoku puzzles. For obvious reasons the larger the grid the more difficult it is to solve the puzzle, but many of the same strategies for solving the traditional 9×9 grids will work when looking at a more general puzzle.

Sudoku Complexity

Many Sudoku authors give ratings to their puzzles. Some of these may be based on computer run time, but often these ratings seem arbitrary. Depending on the algorithm chosen by a person or computer to solve the puzzle, the answer might be found very quickly or it may take much longer. Puzzles that are rated harder aren't necessarily hard because they are more logically challenging, but possibly because the computer algorithm used to find a solution started along a poorly chosen path and therefore had a longer run time. Nevertheless, computers tend to be very fast at solving typical Sudoku puzzles.

In considering the generalized $n^2 \times n^2$ Sudoku grids, the time it takes a computer to reach a solution seems to grow exponentially with n. This results in Sudoku puzzles belonging to a complexity class of problems known as NP-complete. (See `www.imai.is.s.u-tokyo.ac.jp/~yato/data2/SIGAL87-2.pdf`) NP-complete problems have unpredictable running times and the number of solution paths has the potential to become very large and nasty.

Further Research

In addition to the topics already raised, future explorations might include other variants such as Twodoku, determining the minimal number of givens needed for the puzzle to be well-formed, and other combinatorial considerations such as counting the number of well-formed puzzles that can be created from each completed grid. For insight about what is known about these topics and for other possible research questions you can browse James Madison University's Problem of the Week, *Spring 2006: Special All-Sudoku Semester* `www.math.jmu.edu/~taal/POW/`, presented by Professor Laura Taalman, as well as posts on `www.sudoku.com/forums`.

Sudoku a Waste of Time?

In addition to its potential for research, Sudoku, along with other logic puzzles, may prove to be an effective teaching tool within math classrooms. Mathew Mitchell in "Situational Interest: Its Multifaceted Structure in the Secondary School Mathematics Classroom" found that logic puzzles spark the interest of students and make "them really use their brains" (*Journal of Educational Psychology* 3: 424-436). This facilitates understanding of more complex and unusual math problems and helps students develop strategies for problem solving that can be applied to future problems.

Sudoku is much more than simply an entertaining puzzle. When looked at from a mathematical or computer science perspective, further exploration can open new doors and introduce new problems. Sudoku has the potential to lead us to better algorithms for solving hard problems in computer science, to serve as an interesting and approachable platform for undergraduate research in mathematics, and to help more students engage in mathematical thinking. ■

Further Reading

Robin Wilson, "The Sudoku Epidemic," *FOCUS*, Vol. 26.1 (2006), 5-7.