Worksheet: log approximations: Solutions

EC 308

Introduction to logarithms

Practice: basic applications

1. $3^2 = 9$
2. What exponent can you put on 3 to get 9?

   2

3. $\log_3 9 = \ 2$
4. Rearrange this equation with log: $2^3 = 8$.

$$\log_2 8 = 3$$

5. Rearrange this equation to not use log: $\log_{10} 10000 = 4$.

$$10^4 = 10000$$

6. What is $\log_{10}(-1)$?

   Nothing. It does not exist. You cannot take the log of a negative number.

Natural log and e

Practice:

7. Approximately what is the natural log of 7.4? Make an approximate guess without a calculator. Remember that $e \approx 2.718$.

   2.718 to what power equals 7.4?

   $2^2 = 4$ *and* $3^2 = 9$.

   $2.5^2 = 2 \times 2.5 + .5 \times 2.5 = 6.25$.

   A reasonable guess, then, is that 7.4 is the square of something between 2.5 and 3 (like e).

8. Calculate $\ln 7.4$. How does it compare to your guess?

$$ln\, 7.4 \approx 2$$

Graphs

9. Plot $y = e^x$. Describe the shape. What are some variables in the real world that might have this sort of shape?

    Real world variables:

    - *Population (Y) of an area over time (x)*
    - *Average income over time*

10. Remember that logs are exponents. Then log y = x. Given this information, what do you expect a graph of y = log x to look like?

    The two are inverse functions. Thus, the graph should swap x and y in the previous graph. You can get the graph of one by mirroring the other over the line y=x.

11. Make the graph.

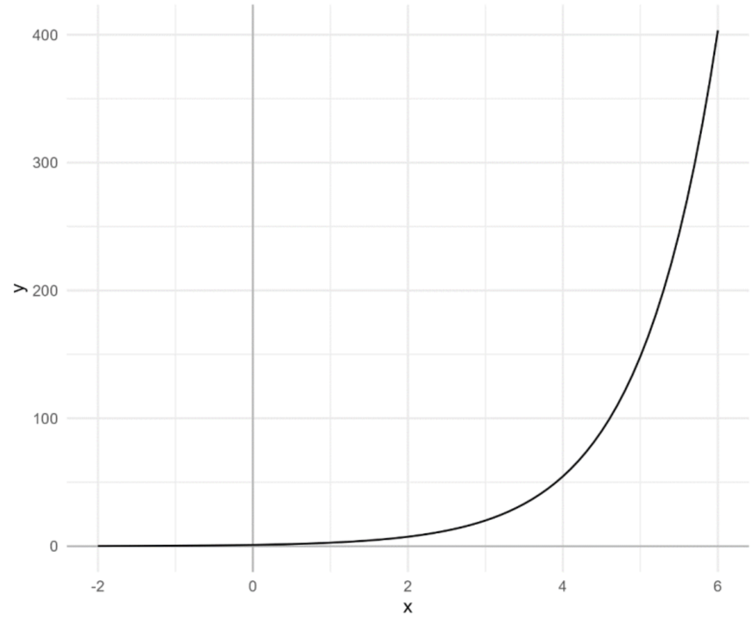

Figure 1: exponential function

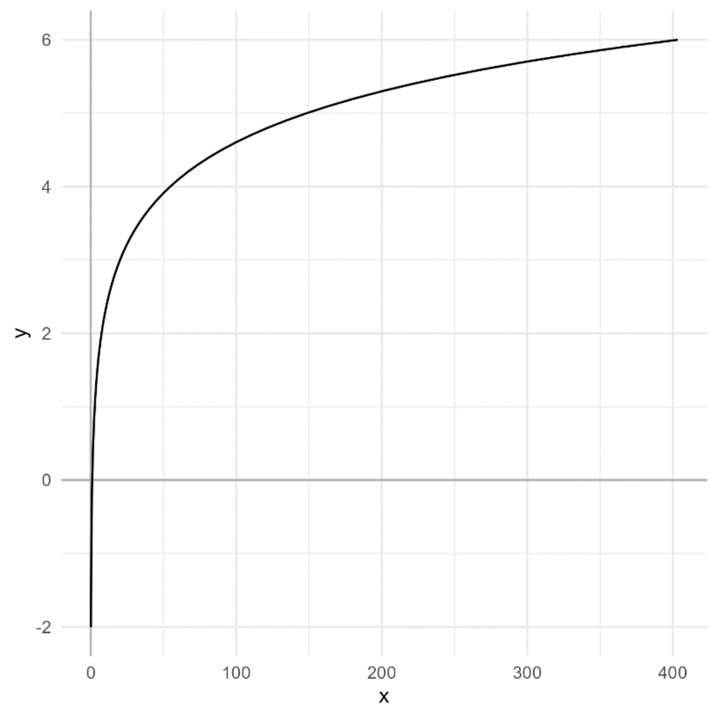

Figure 2: logarithmic function

Log approximations

Practice

12. Practice with simple percent differences

    a. What number is 10% greater than 100?

    $$100 \times 1.10 = 110$$

    b. What number is 10% greater than 20?

    $$20 \times 1.10 = 22$$

    c. What number is 3% greater than 100?

    $$100 \times 1.03 = 103$$

    d. What number is 3% greater than 20?

    1% of 20 is .2. 3% of 20 is .6.

    $$20 \times 1.03 = 20 + .01 \times 20 \times 3 = 20.6$$

13. Suppose that A is 3% greater than B.

   e. Write that relationship as an equation: $A = B \times (1 + .03)$

   f. Take the log of both sides and simplify:

   $$\ln A = \ln(B \times 1.03) = \ln B + \ln 1.03$$

   g. Use the rules above to rewrite this as

   $$\ln A - \ln B = \ln 1.03 \approx .03$$

   h. Finish this sentence: If $\ln A - \ln B = .02$, then A is *approximately 2%* greater than B.

   i. Suppose now that B is 3% larger than C. Write the relationship between A (not B) and C as an equation.

   $$A = B \times 1.03 = C \times 1.03 \times 1.03 = C \times 1.03^2$$

14. Suppose that the average income in an area is related to the average years of education. Each year of school increases earnings by 7% on average (a typical estimate from the econ research on returns to schooling).

   j. Write an equation to relate average earnings Y to average years of school S. Hint: assume there is some average income Y_0 when $S = 0$.

   $$Y_t = Y_0 \times 1.07^S$$

   k. Use logs to rewrite and simplify the equation from part a.

   $$\ln Y_t - \ln Y_0 = \ln 1.07^S = S \times \ln 1.07$$

   Or leave $\ln Y_0$ on the right. It is just some constant value.

   $$ln\, Y_t = S \times ln\, 1.07 + C$$

   This also means that

   $$ln\, Y_t \approx S \times .07 + C$$

   Why this is useful: this is a linear relationship between $ln\, Y$ and S. The slope is $ln\, 1.07 \approx .07$ and the vertical intercept is $C = ln\, Y_0$.

   If we did not know that the effect is 7%, we could get data on Y and S and then regress the log of Y on S to estimate the effect. We would estimate this regression:

   $$ln\, Y = \alpha + \beta S + \varepsilon$$

   β is (approximately) the proportional effect of S on Y. A 1-year increase in S is associated with a $\beta \times 100\%$ increase in Y (approximately).

15. Find estimates of the average incomes for two countries with similar incomes and for two countries with very different incomes. In each case,

    I used data series NY.GDP.PCAP.PP.CD from the World Bank. This is GDP per capita, PPP (current international $). All data are for 2022.

    Note: The difference between two numbers is unambiguous (other than the sign). However, the percent difference depends on which number we use as the denominator. In other words, the percent difference between 10 and 8 is not the same as the percent difference between 8 and 10. In the table, I show the differences in both directions and also include a difference that uses the average of the two incomes as the denominator.

	Income	Proportional difference	Log difference
Similar incomes			
Italy	\$51,865	$\frac{51865-50070}{50070} \approx 0.0359 = 3.59\%$	$\ln 54603 - \ln 50070 \approx 0.0352$
Republic of Korea	\$50,070	$\frac{50070-51865}{51865} \approx -0.0346 = -3.46\%$	Pretty good approximation
		$\frac{51865-50070}{(51865+50070)/2} \approx 0.0352 = 3.52\%$	
Very different incomes			
Mexico	\$21,512	$\frac{21512-2792}{2792} \approx 6.70 = 670\%$	$\ln 21512 - \ln 2792 \approx 2.04$
Rwanda	\$2,792	$\frac{2792-21512}{21512} \approx -0.870 = -87\%$	Bad approximation
		$\frac{21512-2792}{(21512+2792)/2} \approx 1.54 = 154\%$	Better approximation of percent diff with mean in denominator. Still not great.