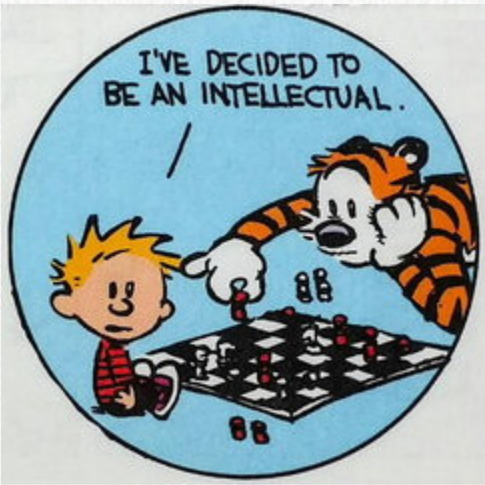


Ampere's law

Let's return to our recent discussions of magnetic fields. The Biot-Savart law for magnetism is analogous to Coulomb's law for electricity. You start with a bunch of individual field sources spread all over the place and add up all their contributions to find the net field ***at one point***. Gauss's law for electric fields describes the field produced by charges in a more general fashion: The electric field ***at a great many points*** determines the net electric flux passing through a surface. That flux is directly proportional to the net electric charge contained within that surface This expresses a general property about the field's gross features rather than focusing on the field at one point in space. The benefit is an enormously general tool that is independent of complicated details about how the charges are configured. The shortcoming is that there are very few situation where it can be used to find the actual field at a point, but the aesthetic beauty of Gauss's law is sufficient to justify its inclusion in your education.

There is another law, similar in spirit to Gauss's, which relates an average of a field at different places to sources of the field distributed through some region of space. Gauss's law for electric fields refers to a closed surface, i.e., one with no edges, so it is the boundary of a certain volume of space. In contrast, Ampere's law for magnetic fields refers to a surface that does have an edge. Basically, you calculate the component of the magnetic field tangent to the edge at every point, then take the average. That value turns out to be proportional to the total amount of current passing through that surface.

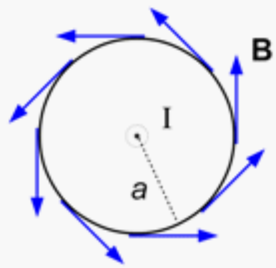
The fluid analogy for electric fields led us to envision electric charges as sources and sinks of electric field lines. That certainly agrees with the picture of a lone point positive charge as having electric fields lines emanating from it in every direction. But magnetic field lines never begin or end, so they don't have a similar source. Instead, magnetic field lines encircle currents. This leads to a different analogy with fluid flow: electric currents determine how much the magnetic field curls around in a mathematically precise fashion, as follows. Calculate the net result of adding up all the tangent parts of field around the circular path by performing the following integration:

$$\int \vec{\mathbf{B}} \cdot d\vec{\boldsymbol{\ell}} = \int B_{tangent} d\ell = \int B \cos\theta \, d\ell$$

The first version is the formal vector notation where ***dl*** is the infinitesimal displacement tangent to the curve, while the second expresses the essential ideas. Borrowing fluid terminology, this expression is called the

circulation of the magnetic field.

To illustrate an application of this formula, consider the magnetic field a distance *a* from a straight wire carrying a current of magnitude *I* . Using the right hand rule, if the current is coming toward you, the field lines encircle the current in a counter-clockwise direction with field vectors tangent to the circle, as shown in this figure:



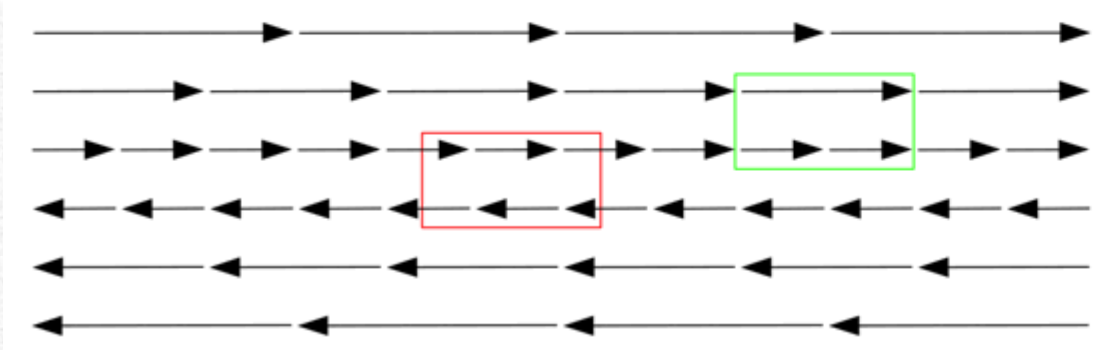
If these vectors depicted velocity in a fluid, we'd describe this as something like a vortex and say the fluid was circulating around the center. While the magnetic field continually changes in direction around the circle, it is always tangent and the magnitude of the tangent part is constant. Hence, the integrand has a constant value *B*, and the remaining integration simply gives the circumference of the circle $2\pi a$. Using the known solution $B = (\mu_0/2\pi)\, I/a$, this means that the circulation can be evaluated without complicated manipulation:

$$\int \vec{\mathbf{B}} \cdot d\vec{\boldsymbol{\ell}} = \int \left[\frac{\mu_0}{2\pi}\frac{I}{a}\right] d\ell = \frac{\mu_0}{2\pi}\frac{I}{a}\int d\ell = \frac{\mu_0}{2\pi}\frac{I}{a}\, 2\pi a = \mu_0 I$$

Surprisingly, this expression is totally independent of what size circle we use. While derived for a specific situation, this simple result turns out to be absolutely general and applies to all steady currents. Moreover, it's true regardless of the shape of the curve we used, and it need not be a circle. The result only depends on how much current passes through the surface bounded by the curve. It can be expressed in a relation called Ampere's law that is analogous to Gauss's law for the total electric flux coming from an electric charge:

- The circulation of magnetic field around the boundary of a surface equals the net amount of current passing through that surface multiplied by μ_0.

This is really neat, but what is it good for? Like Gauss's law for electric fields, there are only a few practical examples where you can directly apply and evaluate Ampere's law to find the magnetic field. As you might guess, they require a strong degree of symmetry, as in the situation above around a line current. However, it is also useful for deducing the current direction from the magnetic field. The figure below illustrates a magnetic field that points horizontally but has no variation in the horizontal direction. However, its magnitude does vary in the vertical direction. It gets stronger pointing to the right as position increases away from the center, and it gets stronger pointing to the left as position decreases from the center.



The colored boxes indicate Amperian loops (the analog to Gaussian surfaces), around which we can calculate the circulation. For the red loop, the vectors are directed rightward on the top and leftward on the bottom, which both contribute to a clockwise circulation. (There is no tangent field on the left and right sides of the loop.) Using the right-hand rule, this tells us that there is a current into the page. For the green loop, the field is rightward on both the top and bottom, but it has a stronger magnitude on the top. Hence, the stronger clockwise contribution on top exceeds the counterclockwise contribution on the bottom, and the net circulation is once again clockwise.

Induction and Faraday's law

Flux is important in situations where we aren't just interested in the value of a field at some point, but rather in its behavior over some extended region. For example, Gauss' law makes powerful statements about what happens when we average the electric and magnetic fields over the surface of some volume. This average is most elegantly expressed in terms of flux:

- The net electric flux through the surface of any volume in space is proportional to the electric charge contained within
- The net magnetic flux through the surface of any volume in space is zero

These surfaces are called closed because they have no edges or, equivalently, you can think of the edges as closing back in on themselves. (A sphere is a closed surface, but a flat disk is not.) If the net amount of flux passing through any closed surface is not zero, we say that this indicates a *divergence* of the electric field.

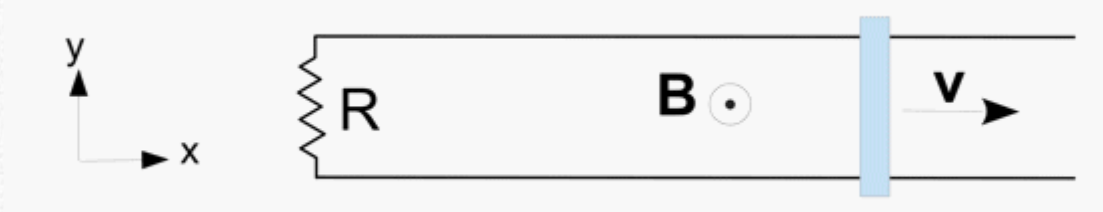
In contrast, Ampere's law deals with surface areas that do have edges, i.e., an open surface. In this case, the quantity to be described in terms of a flux is the electric current.

- The average tangential magnetic field around a loop is proportional to the flux of electric current through it.

It doesn't matter how the current intersects that surface, whether it bunches together in a tightly constricted wire or spreads out in a broad, lazy, unhurried drift. So long as the same total amount of current passes through the surface, that average of the magnetic field will be the same. If we picture magnetic field lines as looking something like streamlines in a fluid, a net forward (or reverse) component of the field reminds one of a whirlpool or some other circulating flow pattern. Hence, we say that the magnetic field has a net *circulation* around the loop.

To motivate our next finding, consider the situation illustrate here. A long pair of metal horizontal rails are in a region with a uniform magnetic field pointed out of the page. Their left ends are connected by a resistor. A conducting rod (blue-grey) lies across the two rails and forms a bridge between them. When the bar is forced

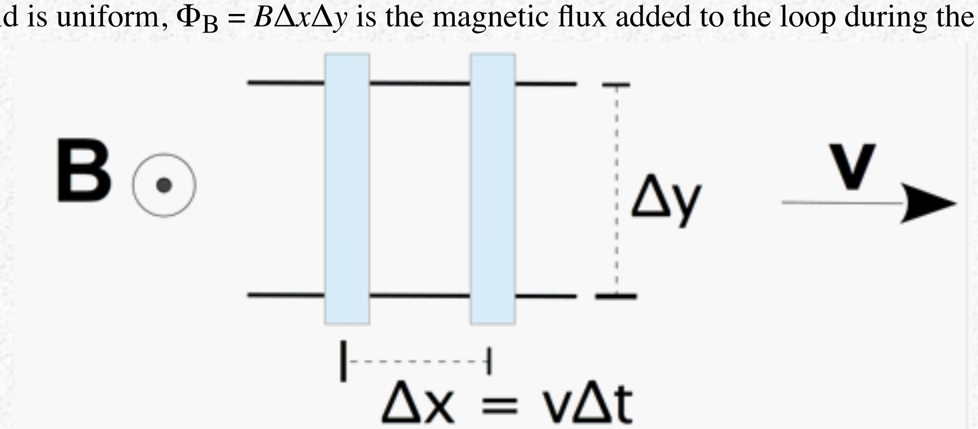
to move rightward at a constant speed v, what happens? The charges in the conducting rod now have a velocity perpendicular to the magnetic field, so they will feel a magnetic force in a direction determined by the right hand rule. For positive particles, the force is downward, though the positive ions that form a metal conductor are immobile. For the negative electrons, the force is upward. The electrons can't complete full circles because they continually run into the ions which are forced to move rightward. Those collisions continually impel the electrons to move rightward as well, and the magnetic force continually nudges them upward. In this way, the electrons gradually make their way toward the top of the conducting bar. (Note that this motion corresponds to a conventional current that is downward in the bar.) At the conducting rails, the current can continue freely to the left end where the two rails are connected, and the circuit is completed across the resistor. The result is that a clockwise conventional current has been induced around the loop. This process is called induction.



Because this configuration drives a current, it necessarily involves an electromotive force (emf). Till now we have pushed off the explanation of emf to the chemistry department or to Alabama Power, but this situation drives a current with an entirely local and non-chemical mechanism. What is actually providing the motive force that energizes the current?

To explore it, note that in the resistor, there is a force on each charged particle of magnitude $F_E = qE$. The magnitude of the electric field is related to the potential drop across the resistor by $\Delta V_{resistor} = E\,\Delta y$, where Δy is the distance along the resistor. Now, the potential difference across the moving bar ΔV_{bar} = emf must have the same value by Kirchoff's loop rule, but the force per particle in the bar has a magnitude $F_B = qvB$. (There is no $\sin\theta$ factor because the velocity and magnetic fields are at right angles to each other.) By analogy between $F_E = qE$ and $F_B = qvB$, the quantity vB plays the same role in the moving bar as E does in the resistor. Therefore, emf = $vB\,\Delta y$ is the electromotive force produced by making the bar move through the magnetic field. The right side is essentially the work per charge, which should harken back to electric potential as the change in electric potential energy per charge.

We can rewrite the emf in an illuminating fashion by noting that the velocity **v** here is in the x direction, so its magnitude is $v = \Delta x/\Delta t$. This means that emf $= vB\,\Delta y = B\,\Delta x\Delta y/\Delta t$. The figure below expands the right end of the above figure. It shows the bar in two positions it occupies at times separated by Δt. During that period the bar moves a distance Δx, so the area enclosed by the total circuit loop increases by $\Delta x\Delta y$. Because the magnetic field is uniform, $\Phi_B = B\Delta x\Delta y$ is the magnetic flux added to the loop during the time Δt.

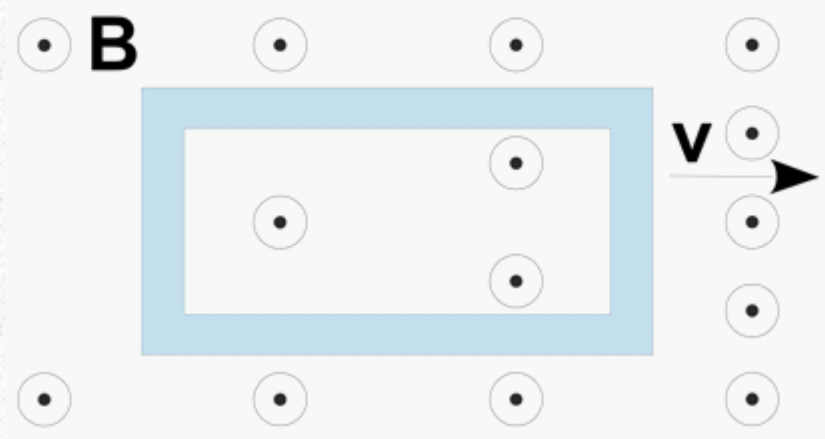


With that identification, the emf can be identified as $\Delta\Phi_B / \Delta t$, i.e., the rate of change of magnetic flux through the circuit with respect to time.

The above might seem like a curious algebraic relation, but further work and a lot of experimentation shows that it's actually a fundamental truth of equal importance to the others listed above, which is called Faraday's law:

- The net electromotive force around any closed loop is equal to the time rate of change of the magnetic flux through the surface bounded by the loop

Here's another example to illustrate the power of this perspective. A hollow loop in the shape of rectangle is dragged through an inhomogeneous magnetic field that gets stronger to the right, perhaps because the source is to the right. The magnetic force per particle is stronger on the right because the field is stronger there than on the left, the result being that a clockwise current will be driven. Because the loop is being pulled into a stronger field, the flux through it is increasing, and sure enough the emf is again given by Faraday's law.



To cap things off, consider the last example from the perspective of an observer who is at moving along with the loop. From their perspective, the loop has no velocity (**v** = 0). Instead, what they see is that the magnetic flux through the loop is getting larger, perhaps because the source of the magnetic field is being moved closer. But because the emf drives a current whose results can be observed (say, by powering a light bulb), this observer must conclude that the emf is the same. Hence, the general statement of Faraday's law is true regardless of whether the change in magnetic flux arises from moving the current loop through a stationary field or keeping the loop stationary and changing the magnetic field in time.

Finally, though the above arguments depended on the magnetic field, it can't fully describe the physical mechanism that produces the electromotive force that supplies energy to the circuit. That follows because magnetic forces can do no work. What has happened is that the time-dependent magnetic field has produced an additional type of force, one that acts oppositely on positive and negative charges, just like the electric field. Once again, a lot of experimentation shows that this new force acts just like the electric force with one exception: an electrostatic electric field has no circulation, but this new one must. Faraday actually discovered that a time-dependent magnetic field produces a new type of electric field, an electrodynamic one, and the emf it produces is equal to its circulation

$$emf = \int \vec{\mathbf{E}} \cdot d\vec{\boldsymbol{\ell}} = -\frac{d\Phi_B}{dt}$$

Lenz's law

The induced current produces its own magnetic flux through the loop. Indeed, if you wrap your right hand around the induced current in the above figures, you'll find that the induced magnetic field is into the page. This means that, while the motion of the loop or the current source is acting to increase the magnetic flux, the induced current is acting to offset that increase. This brings up the final point of interest about induction: the direction of the induced current produces a magnetic field that partially compensates for the change in magnetic flux from the changing external factors. That's a lot of words, so take them one step at a time.

- Induction involves changing the magnetic flux through a loop
- The result is an induced current around the loop, which produces an induced magnetic field
- The addition of the induced magnetic field to the original magnetic field partially reduces the change in magnetic flux that drove induction in the first place

In brief, induction resists changes in magnetic flux. This fact appears in the negative sign that appears in the mathematical version of Faraday's law above. That negative sign actually has its own name, Lenz's law.

Final Summary - Maxwell's Equations

(This final section is for further contemplation but isn't required reading. You will be a better person if you read it.)

Where did this connection between electric and magnetic fields come from? We found before that charges produce electric fields and moving charges produce magnetic fields, but nothing so far has suggested any closer connection between electric and magnetic fields. This implies a much tighter relation, that the two fields can influence each other *even in the absence of charges*. Let me stress how unexpected this was. Electricity and magnetism were recognized as two independent phenomena, much as electricity and gravitation are today. Sure, the fact that charges produce both fields hints at some connection, but charged particles also have mass, so in principle they produce gravitational fields as well. Even today, nobody expects to drop an uncharged apple and discover that the changing gravitational fields produce electric waves. The discovery in the 19th century that electric and magnetic fields are coupled was as profound as any of the great 20th century discoveries like relativity and quantum mechanics.

Ampere's law, Gauss's laws for electric and magnetic fields, and Faraday's law are something very different in spirit from most of the other rules of physics we've encountered. At first glance, they seem quite vague and not at all practical. They relate the average values of various quantities by smearing out all the interesting details. What good is this? After all, aren't practical answers what we're after? Consider a collection of charged particles moving with various and sundry velocities at some instant. Where are they going to be later? Those are the kind of questions we've been interested in.

Yes, this is the practical, applied chore of physics. In the last century, the great conceit of science was that the future of the entire universe was determined by its state at any instant. After all, we have all these incredibly elegant and powerful laws of nature that decide how every particle and field will dance and weave amidst its neighbors. This ambitious program was largely doomed in the twentieth century by quantum mechanics, which introduced an unavoidable level of unpredictability (though still preserving the cherished conservation laws of energy, momentum, angular momentum, and charge). Einstein's theory of relativity eliminated the idea that one could speak of a universally meaningful "instant". Finally, within the last thirty years chaos theory has made the obvious painfully clear: The multifarious details of the universe are so complicated that calculating the destiny of each electron, sparrow, and galaxy would require a level of

precision unattainable to mortals.

But the real beauty of physics lies not in this sort of cosmic bookkeeping. It is in revealing the fundamental truths of nature, the very essence of those physical laws that constrain phenomena throughout our physical universe. I just mentioned several of those above, the conservation laws. Yes, Newton's laws are great for plugging into computers and calculating the detailed trajectories of spacecraft. Using them, flight programmers programmed cameras on Voyager weeks in advance to snap photos of Neptune's moons while whizzing by at thousands of miles per hour, in light so weak that the aperture had to be left open for minutes. That's impressive. But isn't conservation of energy a more remarkable discovery? Here's an abstract quantity, not a tactile substance or a measurable good, that maintains a steady, imperturbable value regardless of what slings and arrows outrageous fortune might throw.

It is this aesthetic appreciation for understanding the laws of nature deeply that really motivates physicists, and the present topic offers a breathtakingly elegant example. Maxwell showed that laws governing **all** electromagnetic phenomena can be presented in an amazingly brief amount of space:

- The electric field divergence depends upon charge density
- The magnetic field divergence is always zero
- The electric field circulation depends upon the variation in magnetic flux
- The magnetic field circulation depends upon the variation in electric flux plus the current flux.

These are Maxwell's equations. They must be augmented with a few universal constants, a couple of algebraic signs, a good understanding of what divergence and circulation mean mathematically, and many years of quiet contemplation and occasional swearing. You're probably itching to get started.